A general study of extremes of stationary tessellations with examples

Nicolas Chenavier*

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Abstract

Let $\mathfrak{m}$ be a random tessellation in $\mathbf{R}^d$, $d \geq 1$, observed in a bounded Borel subset W and $f(\cdot)$ be a measurable function defined on the set of convex bodies. A point $z(C)$, called the nucleus of C, is associated with each cell C of $\mathfrak{m}$. Applying $f(\cdot)$ to all the cells of $\mathfrak{m}$, we investigate the order statistics of $f(C)$ over all cells $C \in \mathfrak{m}$ with nucleus in $\mathbf{W}_\rho = \rho^{1/d}W$ when ρ goes to infinity. Under a strong mixing property and a local condition on $\mathfrak{m}$ and $f(\cdot)$, we show a general theorem which reduces the study of the order statistics to the random variable $f(\mathcal{C})$, where $\mathcal{C}$ is the typical cell of $\mathfrak{m}$. The proof is deduced from a Poisson approximation on a dependency graph via the Chen-Stein method. We obtain that the point process $\left\{(\rho^{-1/d}z(C), a_\rho^{-1}(f(C)-b_\rho)), C \in \mathfrak{m}, z(C) \in \mathbf{W}_\rho\right\}$, where $a_\rho > 0$ and b_ρ are two suitable functions depending on ρ, converges to a non-homogeneous Poisson point process. Several applications of the general theorem are derived in the particular setting of Poisson-Voronoi and Poisson-Delaunay tessellations and for different functions $f(\cdot)$ such as the inradius, the circumradius, the area, the volume of the Voronoi flower and the distance to the farthest neighbor.

Keywords: Random tessellations; extreme values; order statistics; dependency graph; Poisson approximation; Voronoi flower; Poisson point process; Gauss-Poisson point process.

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1 Introduction

A tessellation of $\mathbf{R}^d$, $d \geq 1$, endowed with its Euclidean norm $|\cdot|$, is a countable collection of nonempty compact subsets, called *cells*, with disjoint interiors which subdivides the space and such that the number of cells intersecting any bounded subset of $\mathbf{R}^d$ is finite. The set $\mathbf{T}$ of tessellations is endowed with the σ-algebra generated by the sets $\left\{m \in \mathbf{T}, \bigcup_{C\in m} \partial C \cap K = \varnothing\right\}$ where ∂K is the boundary of K for any compact set K in $\mathbf{R}^d$. By a random tessellation $\mathfrak{m}$, we mean a random variable with values in $\mathbf{T}$. It is said to be stationary if its distribution is invariant under translations of the cells. For a complete account on random tessellations, we refer to the books [35, 40] and the survey [7].

Given a fixed realization of $\mathfrak{m}$, we associate with each cell $C \in \mathfrak{m}$ in a deterministic way a point $z(C)$, which is called the *nucleus* of the cell, such that $z(C+x) = z(C) + x$ for all $x \in \mathbf{R}^d$. To describe the mean behaviour of the tessellation, the notions of intensity and typical cell are introduced as follows. Let B be a Borel subset of $\mathbf{R}^d$ such that $\lambda_d(B) \in (0,\infty)$, where λ_d is the d-dimensional Lebesgue measure. The *intensity* γ of the tessellation is defined as $\gamma = \frac{1}{\lambda_d(B)} \cdot \mathbb{E}\left[\#\{C \in \mathfrak{m}, z(C) \in B\}\right]$ and we assume that $\gamma \in (0,\infty)$. Since $\mathfrak{m}$ is stationary, γ is independent

*Postal address: Université de Rouen, LMRS, avenue de l'Université, BP 12 76801 Saint-Etienne-du-Rouvray cedex, France. E-mail: nicolas.chenavier@etu.univ-rouen.fr

of B and we suppose, without loss of generality, that $\gamma = 1$. The *typical cell* $\mathcal{C}$ is a random polytope whose the distribution is given by

$$\mathbb{E}[f(\mathcal{C})] = \frac{1}{\lambda_d(B)} \cdot \mathbb{E}\left[\sum_{\substack{C\in\mathfrak{m},\\ z(C)\in B}} f(C - z(C))\right] \tag{1.1}$$

for all $f : \mathcal{K}_d \to \mathbf{R}$ bounded measurable function on the set of convex bodies $\mathcal{K}_d$, i.e. convex compact sets, endowed with the Hausdorff topology.

We are interested in the following problem: only a part of the tessellation is observed in the window $\mathbf{W}_\rho = \rho^{1/d}W$, where W is a bounded Borel subset of $\mathbf{R}^d$, i.e. included in a cube $\mathbf{C}^{(W)}$, and such that $\lambda_d(W) \neq 0$. Let $f : \mathcal{K}_d \to \mathbf{R}$ be a translation invariant measurable function, i.e. $f(C + x) = f(C)$ for all $C \in \mathcal{K}_d$ and $x \in \mathbf{R}^d$. For all $r \in \mathbf{N}^*$, we denote by $M^{(r)}_{f,\mathbf{W}_\rho}$ the r-th order statistic of f over the cells $C \in \mathfrak{m}$ such that $z(C) \in \mathbf{W}_\rho$. We have chosen the convention to call the r order statistics the r largest values. When $r = 1$, the 1-st order statistic is denoted by $M_{f,\mathbf{W}_\rho}$, i.e.

$$M_{f,\mathbf{W}_\rho} = M^{(1)}_{f,\mathbf{W}_\rho} = \max_{\substack{C\in\mathfrak{m},\\ z(C)\in\mathbf{W}_\rho}} f(C).$$

In this paper, we investigate the limit behaviour of $M^{(r)}_{f,\mathbf{W}_\rho}$ when ρ goes to infinity.

The study of extremes could describe the regularity of the tessellation (e.g. presence of elongated cells). For instance, in finite element method, the quality of the approximation depends on some consistency measurements over the partition, see e.g. [14]. Another potential application field is statistics of point processes. The key idea would be to identify a point process from the extremes of a tessellation induced by the point process.

To the best of our knowledge, one of the first works on extreme values in stochastic geometry is due to Penrose. In chapters 6,7 and 8 in [26], he investigates the maximum and minimum degrees of random geometric graphs. More recently, Schulte and Thäle [36] establish a theorem to derive the order statistics of a functional $f_k(x_1, \dots, x_k)$ of k points on a homogeneous Poisson point process. Nevertheless, their approach cannot be applied to our problem. Indeed, studying extremes of the tessellation requires to use functionals which depend on the whole point process of nuclei and not only on a fixed number of points. Besides, we consider any translation invariant measurable function $f(\cdot)$ and we restrict our investigation to a certain kind of random tessellation satisfying a strong mixing property (Condition (FRC), p. 3). Our general theorem (Theorem 1, p. 3) is followed by numerous examples, with rates of convergence, in the particular setting of Poisson-Voronoi and Poisson-Delaunay tessellations. This improves in particular some extremes that are investigated in [8]. Before stating our main theorems, we need some preliminaries which contain notation and conditions on the random tessellation.

Preliminaries. Let $\mathbf{C}^{(W)}$ be a cube in $\mathbf{R}^d$ containing W. We partition $\mathbf{C}^{(W)}_\rho = \rho^{1/d}\mathbf{C}^{(W)}$ into a set V_ρ of N_ρ sub-cubes of equal size, such that $N_\rho^{1/d}$ is an integer and $N_\rho \underset{\rho\to\infty}{\longrightarrow} \infty$. These sub-cubes are indexed by the set of $\mathbf{i} = (i_1, \dots, i_d) \in \left[1, N_\rho^{1/d}\right]^d$. With a slight abuse of notation, we identify a cube with its index. Let us define a distance between sub-cubes $\mathbf{i}$ and $\mathbf{j}$ as $d(\mathbf{i}, \mathbf{j}) = \max_{1\le r\le d}\{|i_r - j_r|\}$. Moreover, if A, B are two sets of sub-cubes, we let $d(A, B) = \min_{\mathbf{i}\in A,\mathbf{j}\in B} d(\mathbf{i}, \mathbf{j})$. For each $\mathbf{i} \in V_\rho$, we denote by

$$M_{f,\mathbf{i}} = \max_{\substack{C\in\mathfrak{m},\\ z(C)\in\mathbf{i}\cap\mathbf{W}_\rho}} f(C). \tag{1.2}$$

When $\{C \in \mathfrak{m}, z(C) \in \mathbf{i} \cap \mathbf{W}_\rho\}$ is empty, we take $M_{f,\mathbf{i}} = -\infty$.

Let us consider a threshold v_ρ depending on ρ. Studying the order statistics amounts to investigate the number of exceedance cells of v_ρ denoted by

$$U_\rho(v_\rho) = \sum_{\substack{C\in\mathfrak{m},\\ z(C)\in\mathbf{W}_\rho}} \mathbb{1}_{f(C)>v_\rho}. \tag{1.3}$$

Since f is translation invariant, the mean of this random variable is, according to (1.1):

$$\mathbb{E}\left[U_\rho(v_\rho)\right] = \lambda_d(\mathbf{W}_\rho)\cdot\mathbb{P}\left(f(\mathcal{C})>v_\rho\right). \tag{1.4}$$

We assume the following condition which is referred as the typical cell property (TCP):

Condition (TCP): *the mean number of exceedance cells converges to a limit denoted by $\tau\geq 0$, i.e.*

$$\lambda_d(\mathbf{W}_\rho)\cdot\mathbb{P}\left(f(\mathcal{C})>v_\rho\right)\underset{\rho\to\infty}{\longrightarrow}\tau.$$

Moreover, we denote by $G_1(\rho)$ the rate of convergence, i.e.

$$G_1(\rho) = \left|\lambda_d(\mathbf{W}_\rho)\cdot\mathbb{P}(f(\mathcal{C})>v_\rho)-\tau\right|. \tag{1.5}$$

We assume also a (global) condition of R-dependence associated with $\mathfrak{m}$ and f which is referred as the finite range condition (FRC).

Condition (FRC): *there exists an integer R and an event A_ρ with $\mathbb{P}\left(A_\rho\right)\underset{\rho\to\infty}{\longrightarrow}1$ such that, conditional on A_ρ, the σ-algebras $\sigma\{M_{f,\mathbf{i}},\mathbf{i}\in A\}$ and $\sigma\{M_{f,\mathbf{i}},\mathbf{i}\in B\}$ are independent when $d(A,B)>R$.*

Finally, in order to present our first theorem, we introduce a second function defined as

$$G_2(\rho) = N_\rho\mathbb{E}\left[\sum_{\substack{(C_1,C_2)_{\neq}\in\mathfrak{m}^2,\\ z(C_1),z(C_2)\in\mathfrak{C}_\rho}} \mathbb{1}_{f(C_1)>v_\rho,f(C_2)>v_\rho}\right], \tag{1.6}$$

where

$$\mathfrak{C}_\rho = \left[0,(2R+1)\cdot\lambda_d(\mathbf{W}_\rho)^{1/d}N_\rho^{-1/d}\right]^d \tag{1.7}$$

and where $(C_1,C_2)_{\neq}\in\mathfrak{m}^2$ means that (C_1,C_2) is a pair of distinct cells of $\mathfrak{m}$.

Order statistics. We are now prepared to present our first theorem.

Theorem 1. *Let $\mathfrak{m}$ be a stationary random tessellation of intensity 1 such that* Conditions (TCP) *and* (FRC) *hold and let $r\in\mathbf{N}^*$ be fixed. Then*

$$\left|\mathbb{P}(M_{f,\mathbf{W}_\rho}^{(r)}\leq v_\rho) - e^{-\tau}\sum_{k=0}^{r-1}\frac{\tau^k}{k!}\right| = O\left(G_1(\rho)+G_2(\rho)+N_\rho^{-1}+\mathbb{P}(A_\rho^c)\right), \tag{1.8}$$

where $\phi(\rho)=O(\psi(\rho))$ means that $\phi(\rho)/\psi(\rho)$ is bounded as ρ goes to infinity.

To derive useful applications, we assume a second condition on the random tessellation referred as the local correlation condition (LCC).

CONDITION (LCC): *the function $G_2(\rho)$ converges to 0 as ρ goes to infinity.*

This (local) condition means that with high probability two neighboring cells are not simultaneously exceedances. Under this assumption, we obtain the following result:

Corollary 1. *Let $\mathfrak{m}$ be a stationary random tessellation of intensity* 1 *such that* CONDITIONS (TCP), (FRC) *and* (LCC) *hold and let $r \in \mathbf{N}^*$ be fixed. Then*

$$\mathbb{P}(M_{f,\mathbf{W}_\rho}^{(r)} \leq v_\rho) \underset{\rho\to\infty}{\longrightarrow} e^{-\tau}\sum_{k=0}^{r-1}\frac{\tau^k}{k!}$$

with rate of convergence $O\left(G_1(\rho) + G_2(\rho) + N_\rho^{-1} + \mathbb{P}(A_\rho^c)\right)$.

Remark 1. When CONDITION (LCC) does not hold, we can show that $\lim_{\rho\to\infty}\mathbb{P}\left(M_{f,\mathbf{W}_\rho} \leq v_\rho\right) = e^{-\theta\tau}$ for some constant $\theta \in [0,1]$, provided that the limit of $\mathbb{P}\left(M_{f,\mathbf{W}_\rho} \leq v_\rho\right)$ exists. Its proof relies notably on the adaptation to our setting of several arguments included in [17]. According to Leadbetter, we say that θ is the *extremal index.* In a future work, we hope to develop a general method to estimate this quantity.

Theorem 1 could be extended in several directions. First, its result remains true for random tessellations satisfying some β-mixing property (see examples of Voronoi tessellations given in [11]). Moreover, it can be generalized to marked point processes satisfying three properties: two conditions similar to CONDITIONS (FRC) and (LCC), and one condition on the mark distribution. In particular, we could investigate extremes of Boolean models with bounded grains. When the random tessellation is ergodic with respect to the group of translations of $\mathbf{R}^d$, the order statistics are asymptotically independent of the choice of nuclei $z(\cdot)$. Indeed, they only depend on the asymptotic behaviour of $G_1(\rho)$ and the typical cell $\mathcal{C}$ itself does not depend on the set of nuclei thanks to the ergodic theorem (see [9], p. 339). This will be the case for examples that we deal with. Moreover, we notice that the order statistics do not depend on the shape of the window W. Actually, a method similar to Proposition 3 of [8] shows that the contribution of boundary cells is negligible. Besides, we do not know if our rate of convergence is optimal. The technique that we use is similar to the method employed by Penrose and Yukich [27]. Nevertheless, their rate of convergence is sub-optimal (e.g. better estimates can be found in [15]). So we suppose that the improvement of the results might be achieved by using other techniques, different from ours.

As mentioned above, CONDITIONS (FRC) and (LCC) concern global and local properties of the tessellation respectively. In fact, there exists an analogy between CONDITIONS (FRC) and (LCC) and Conditions $D(u_n)$ and $D'(u_n)$ of Leadbetter [16] respectively. The general theory of extreme values deals with sequences [12] or random fields [19], see also the reference books [10] and [32]. Unfortunately, we are unable to apply it in our setting. Indeed, the set of random variables that we consider is not a discrete random field in a classical meaning. More precisely, the process $\{M_{f,\mathbf{i}}\}_{\mathbf{i}\in V_\rho}$ is a triangular array indexed by $\mathbf{N}^d$ and the process $\{f(C_x)\}_{x\in\mathbf{R}^d}$ is not a Gaussian continuous random field, where C_x is the cell of the tessellation containing x.

Point process of exceedances. Concretely, the threshold is often of the form $v_\rho = v_\rho(t) = a_\rho t + b_\rho$, $t \in \mathbf{R}$ with $a_\rho > 0$. In this case, we can be more specific about the joint distributions of the order statistics. Before stating our second theorem, we need some preliminaries. We denote by $\tau(t) \in [0,+\infty]$, $t \in \mathbf{R}$, the limit of $\lambda_d(\mathbf{W}_\rho)\cdot\mathbb{P}(f(\mathcal{C}) > v_\rho(t))$ as ρ goes to infinity and by ${}_*x = \inf\{t \in \mathbf{R}, \tau(t) < \infty\}$ and $x^* = \sup\{t \in \mathbf{R}, \tau(t) > 0\}$ the lower and upper endpoints of $\tau(\cdot)$. Since a_ρ is positive, the function $\tau(\cdot)$ is not increasing so that $\tau(\cdot)$ is finite on $({}_*x, x^*]$.

Under CONDITIONS (FRC) and (LCC), we consider the random collection

$$\Phi_\rho = \left\{\left(\rho^{-1/d}z(C), a_\rho^{-1}(f(C) - b_\rho)\right), C \in \mathfrak{m} \text{ and } z(C) \in \mathbf{W}_\rho\right\} \subset W \times \mathbf{R}.$$

Moreover, we consider a Poisson point process $\Phi \subset W \times ({}_*x, x^*]$, with intensity measure ν given by

$$\nu(B \times (s,t]) = \mathbb{E}\left[\#\Phi \cap (B \times (s,t])\right] = \frac{\lambda_d(B)}{\lambda_d(W)} \cdot (\tau(s) - \tau(t))$$

for all Borel subset $B \subset W$ and all segment $(s,t] \subset ({}_*x, x^*]$. We then obtain the following limit theorem.

Theorem 2. *Let* $\mathfrak{m}$ *be a stationary random tessellation of intensity* 1 *such that* Conditions (TCP), (FRC) *and* (LCC) *hold for each* $v_\rho = v_\rho(t) = a_\rho t + b_\rho$, $t \in \mathbf{R}$. *Then the family of point processes* Φ_ρ *converges in distribution to the Poisson point process* Φ, *i.e. for any Borel subsets* $\mathscr{B}_1, \ldots, \mathscr{B}_k \subset W \times ({}_*x, x^*]$ *with* $\nu(\partial\mathscr{B}_i) = 0$, $1 \leq i \leq k$

$$(\#\Phi_\rho \cap \mathscr{B}_1, \ldots, \#\Phi_\rho \cap \mathscr{B}_k) \xrightarrow{\mathcal{D}} (\#\Phi \cap \mathscr{B}_1, \ldots, \#\Phi \cap \mathscr{B}_k),$$

where $\partial\mathscr{B}_i$ *denotes the boundary of* $\mathscr{B}_i$.

This result suggests that the largest order statistics can be seen as points of a (non homogeneous) Poisson point process. Theorem 2 gives their joint distributions so that Corollary 1 is a particular case of the latter when $k = 1$ and $\mathscr{B} = W \times (t, x^*]$. For a wider panorama on results of the point process of exceedances associated with the extremes of a sequence of non independent random variables, we refer to chapter 5 in [18]. When $W = \mathbf{C}^{(W)} = [0,1]^d$ and when $\tau(\cdot)$ is not constant, the function $\tau(\cdot)$ belongs to either the Fréchet, the Gumbel or the Weibull family. This fact is a rewriting of the proof of Theorem 4.1 in [19].

The paper is organized as follows. In Section 2, we show how to reduce our problem to the study of extreme values on a dependency graph. We use a result of [2] to derive an estimate of exceedances by a Poisson distribution. We then deduce Theorems 1 and 2 from a discretization of W into sub-cubes. Sections 3, 4 and 5 are devoted to numerous applications on Delaunay and Voronoi random tessellations. We investigate the asymptotic behaviours with the rates of convergence of:

- the minimum of circumradii of a Poisson-Delaunay tessellation in any dimension and the maximum and minimum of the areas in the planar case (Section 3),
- the minimum of distances to the farthest neighboring nucleus and the minimum of the volume of flowers for a Poisson-Voronoi tessellation (Section 4),
- the maximum of inradii for a Voronoi tessellation induced by a Gauss-Poisson process (Section 5).

For each tessellation and each characteristic, we need to find a suitable threshold v_ρ and to check Condition (LCC) which requires some delicate geometric estimates.

In the rest of the paper, c or c' denotes a generic constant which does not depend on ρ but may depend on other quantities. The term $v_\rho = v_\rho(t)$ denotes a generic function of t, depending on ρ, which is specified in Sections 3, 4 and 5.

2 Proofs of Theorems 1 and 2

2.1 Extreme values on a dependency graph and proof of Theorem 1

We first outline the methodology of the proof of Theorem 1 with some additional notation. A classical method in extreme value theory is to investigate the exceedances. We consider two random variables that are the number of exceedance cells $U_\rho(v_\rho)$, introduced in (1.3), and the number of exceedance cubes $U'_{V_\rho}(v_\rho)$ defined as

$$U'_{V_\rho}(v_\rho) = \sum_{\mathbf{i} \in V_\rho} \mathbb{1}_{M_{f,\mathbf{i}} > v_\rho}, \tag{2.1}$$

where V_ρ and $M_{f,\mathbf{i}}$ are introduced in the preliminaries, p. 2. We denote by μ_ρ the mean of $U'_{V_\rho}(v_\rho)$, i.e.

$$\mu_\rho = \mathbb{E}\left[U'_{V_\rho}(v_\rho)\right] = \sum_{\mathbf{i}\in V_\rho} \mathbb{P}\left(M_{f,\mathbf{i}} > v_\rho\right). \tag{2.2}$$

The proof of Theorem 1 is divided into the three following lemmas.

Lemma 1. *Under the same assumptions as in Theorem 1, we get for all* $r \in \mathbf{N}^*$

$$\left|\mathbb{P}\left(U_\rho(v_\rho) \leq r-1\right) - \mathbb{P}\left(U'_{V_\rho}(v_\rho) \leq r-1\right)\right| \leq 2\cdot G_2(\rho). \tag{2.3}$$

The above lemma is a consequence of Condition (LCC).

Lemma 2. *Let* μ_ρ *be as in* (2.2)*. Under the hypothesis of Theorem 1, we get for all* $r \in \mathbf{N}^*$

$$\left|\mathbb{P}(U'_{V_\rho}(v_\rho) \leq r-1) - e^{-\mu_\rho}\sum_{k=0}^{r-1}\frac{\mu_\rho^k}{k!}\right| = O\left(G_2(\rho) + N_\rho^{-1} + \mathbb{P}(A_\rho^c)\right). \tag{2.4}$$

The derivation of Lemma 2 constitutes the major part of the proof of Theorem 1. It means that the number of exceedance cubes is approximately a Poisson random variable. The fundamental concept to prove this lemma is that of a dependency graph. We first establish a Poisson approximation on the number of exceedances on such graph and we show how our problem can be reduced to this setting. Finally, the following result gives an estimate for μ_ρ.

Lemma 3. *Let* μ_ρ *as in* (2.2)*. Under the assumptions as in Theorem 1, we get*

$$|\mu_\rho - \tau| \leq G_1(\rho) + G_2(\rho).$$

Proof of Theorem 1. Since $M^{(r)}_{f,\mathbf{W}_\rho}$ is lower than v_ρ if and only if $U_\rho(v_\rho) \leq r-1$, we deduce Theorem 1 from the three lemmas above and the fact that the function $x \mapsto e^{-x}\sum_{k=0}^{r-1}\frac{x^k}{k!}$ is Lipschitz. □

In the rest of this subsection, we proceed as follows. We first establish the Poisson approximation on a dependency graph (Proposition 3) and we deduce from it Lemma 2. The key idea is to apply Conditions (TCP) and (FRC). Then we prove Lemmas 1 and 3.

Extreme values on a dependency graph. Given a collection of real random variables $X_\mathbf{i}, \mathbf{i} \in V$ (not necessarily stationary), we say that the graph $G = (V, E)$ is a *dependency graph* for $X_\mathbf{i}$ if for any pair of disjoint sets $A_1, A_2 \subset V$ such that no edge in E has one endpoint in A_1 and the other in A_2, the σ-fields $\sigma(X_\mathbf{i}, \mathbf{i} \in A_1)$ and $\sigma(X_\mathbf{i}, \mathbf{i} \in A_2)$ are mutually independent. Introduced by Petrovskaya and Leontovitch in [28], this concept was applied by Baldi and Rinott (e.g. [5]) to obtain central limit theorems and normal approximations. Furthermore, Arratia *et al.* give a Poisson approximation of a sum of (non independent) Bernoulli random variables indexed by a countable set (see Theorem 1 in [2]). We write their result in our context to approximate the number of exceedances on a dependency graph by a Poisson random variable.

We denote by $|V|$, D and $J \subset \mathbf{R}$, the number of vertices of G, its maximal degree and a finite union of disjoint intervals respectively. Let $\mathbf{U}'_V(J)$ be

$$\mathbf{U}'_V(J) = \sum_{\mathbf{i}\in V} \mathbb{1}_{X_\mathbf{i}\in J}$$

and $p_{\mathbf{i}} = \mathbb{P}(X_{\mathbf{i}} \in J)$, $p_{\mathbf{ij}} = \mathbb{P}(X_{\mathbf{i}} \in J, X_{\mathbf{j}} \in J)$ for all $\mathbf{i} \in V$ and $\mathbf{j} \in V(\mathbf{i}) - \{\mathbf{i}\}$, where $V(\mathbf{i})$ is the set of neighbors of $\mathbf{i}$, i.e.

$$V(\mathbf{i}) = \{\mathbf{j} \in V, (\mathbf{i}, \mathbf{j}) \in E\} \cup \{\mathbf{i}\}.$$

Let us consider a Poisson random variable Z of mean

$$\boldsymbol{\mu}_J = \mathbb{E}[Z] = \mathbb{E}[\mathbf{U}'_V(J)] = \sum_{\mathbf{i} \in V} \mathbb{P}\left(X_{\mathbf{i}} \in J\right).$$

The Chen-Stein method can be applied to approximate the number of occurrences of dependent events by a Poisson random variable (e.g. [2]). In particular, this is a powerful tool to derive some results in extreme value theory for a sequence of real random variables (e.g. [38]). We write below a slightly modified version of Theorem 1 of [2] to derive an upper bound for the total variation distance between $\mathbf{U}'_V(J)$ and its Poisson approximation Z for a dependency graph.

Proposition 3. *(Arratia et al. 1990) Let $p(V) = \sup_{\mathbf{i} \in V} p_{\mathbf{i}}$ and $q(V)^2 = \sup_{(\mathbf{i},\mathbf{j}) \in E} p_{\mathbf{ij}}$. Then*

$$\sup_{A \subset \mathbf{N}} \left|\mathbb{P}(\mathbf{U}'_V(J) \in A) - \mathbb{P}(Z \in A)\right| \leq 2D \cdot |V| \cdot \left(p(V)^2 + q(V)^2\right). \tag{2.5}$$

In particular, for all $r \in \mathbf{N}^*$, we get

$$\left|\mathbb{P}(\mathbf{U}'_V(J) \leq r-1) - e^{-\boldsymbol{\mu}_J} \sum_{k=0}^{r-1} \frac{\boldsymbol{\mu}_J^k}{k!}\right| \leq 2D \cdot |V| \cdot \left(p(V)^2 + q(V)^2\right). \tag{2.6}$$

Proof of Proposition 3. The upper bound (2.6) is a direct consequence of (2.5). From Theorem 1 of [2], we get

$$\sup_{A \subset \mathbf{N}} \left|\mathbb{P}(\mathbf{U}'_V(J) \in A) - \mathbb{P}(Z \in A)\right| \leq 2(b_1 + b_2 + b_3), \tag{2.7}$$

where

$$b_1 = \sum_{\mathbf{i} \in V} \sum_{\mathbf{j} \in V(\mathbf{i})} p_{\mathbf{i}} p_{\mathbf{j}}, \quad b_2 = \sum_{\mathbf{i} \in V} \sum_{\mathbf{i} \neq \mathbf{j} \in V(\mathbf{i})} p_{\mathbf{ij}} \text{ and } b_3 = \sum_{\mathbf{i} \in V} \mathbb{E}\left[\left|\mathbb{P}\left(X_{\mathbf{i}} \in J \middle| \sigma(X_{\mathbf{j}} : \mathbf{j} \notin V(\mathbf{i}))\right) - p_{\mathbf{i}}\right|\right].$$

Since $|V(\mathbf{i})| \leq D+1$, we obtain $b_1 \leq |V| \cdot D \cdot p(V)^2$ and $b_2 \leq |V| \cdot D \cdot q(V)^2$. Moreover, using the fact that if $\mathbf{j} \notin V(\mathbf{i})$, the random variable $X_{\mathbf{j}}$ is independent of $X_{\mathbf{i}}$, we get $b_3 = 0$. We then deduce (2.5) from (2.7). □

The technique of dependency graphs, combined with a discretization technique, has been used on several occasions to derive central limit theorems in geometric probability (see e.g. [3]). In the same spirit, we derive Lemma 2 from Proposition 3. We need first to explain how we construct the dependency graph from our random tessellation.

Construction of the dependency graph. We define a graph $G_\rho = (V_\rho, E_\rho)$ as follows. The set V_ρ consists of the sub-cubes $\mathbf{i}$ ($|V_\rho| = N_\rho$) which cover $\mathbf{W}_\rho$ and an edge $(\mathbf{i}, \mathbf{j}) \in E_\rho$ if $d(\mathbf{i}, \mathbf{j}) \leq R$, where R is introduced in Condition (FRC). The maximal degree D_ρ of this graph satisfies

$$D_\rho \leq (2R+1)^d. \tag{2.8}$$

For all $\mathbf{i} \in V_\rho$, we define the random variable $X_{\mathbf{i}}$ as $X_{\mathbf{i}} = M_{f,\mathbf{i}}$. From Condition (FRC), conditional on A_ρ, the graph G_ρ is a dependency graph for $(M_{f,\mathbf{i}})_{\mathbf{i} \in V_\rho}$.

Proof of Lemma 2. We apply Proposition 3 to $X_{\mathbf{i}} = M_{f,\mathbf{i}}$ and $J = (v_\rho, \infty)$. It is enough to derive upper bounds for $\mathbb{P}(M_{f,\mathbf{i}} > v_\rho | A_\rho)$ and $\mathbb{P}(M_{f,\mathbf{i}} > v_\rho, M_{f,\mathbf{j}} > v_\rho | A_\rho)$. According to (1.2), we get

$$\mathbb{P}(M_{f,\mathbf{i}} > v_\rho) = \mathbb{P}\left(\bigcup_{\substack{C \in \mathfrak{m}, \\ z(C) \in \mathbf{i} \cap \mathbf{W}_\rho}} \{f(C) > v_\rho\} \right) \leq \mathbb{E}\left[\sum_{\substack{C \in \mathfrak{m}, \\ z(C) \in \mathbf{i} \cap \mathbf{W}_\rho}} \mathbb{1}_{f(C) > v_\rho} \right].$$

Since f is translation invariant and $\lambda_d(\mathbf{i}) = \frac{1}{N_\rho} \lambda_d\left(\mathbf{C}_\rho^{(W)}\right) = c(W) \cdot \frac{1}{N_\rho} \lambda_d(\mathbf{W}_\rho)$, where $c(W) = \frac{\lambda_d\left(\mathbf{C}^{(W)}\right)}{\lambda_d(W)}$, we deduce from (1.1) that

$$\mathbb{P}(M_{f,\mathbf{i}} > v_\rho) \leq c(W) \cdot \frac{1}{N_\rho} \lambda_d(\mathbf{W}_\rho) \cdot \mathbb{P}(f(\mathcal{C}) > v_\rho). \tag{2.9}$$

Using the inequalities $\mathbb{P}(M_{f,\mathbf{i}} > v_\rho | A_\rho) \leq \mathbb{P}(M_{f,\mathbf{i}} > v_\rho) / \mathbb{P}(A_\rho)$ and $\lambda_d(\mathbf{W}_\rho) \cdot \mathbb{P}(f(\mathcal{C}) > v_\rho) \leq G_1(\rho) + \tau$, where $G_1(\rho)$ is defined in (1.5), we obtain

$$p_{\mathbf{i}} := \mathbb{P}(M_{f,\mathbf{i}} > v_\rho | A_\rho) \leq c(W) \cdot \frac{G_1(\rho) + \tau}{\mathbb{P}(A_\rho) N_\rho}. \tag{2.10}$$

Moreover, for any $\mathbf{i} \in V_\rho$ and $\mathbf{j} \in V_\rho(\mathbf{i}) - \{\mathbf{i}\}$, we get

$$\begin{aligned} \mathbb{P}(M_{f,\mathbf{i}} > v_\rho, M_{f,\mathbf{j}} > v_\rho) &= \mathbb{P}\left(\bigcup_{\substack{C_1 \in \mathfrak{m}, \\ z(C_1) \in \mathbf{i} \cap \mathbf{W}_\rho}} \bigcup_{\substack{C_2 \in \mathfrak{m}, \\ z(C_2) \in \mathbf{j} \cap \mathbf{W}_\rho}} \{f(C_1) > v_\rho, f(C_2) > v_\rho\} \right) \\ &\leq \mathbb{E}\left[\sum_{\substack{(C_1, C_2)_{\neq} \in \mathfrak{m}^2 \\ z(C_1), z(C_2) \in V_\rho(\mathbf{i})}} \mathbb{1}_{f(C_1) > v_\rho, f(C_2) > v_\rho} \right], \end{aligned} \tag{2.11}$$

where $(C_1, C_2)_{\neq} \in \mathfrak{m}^2$ means that (C_1, C_2) is a pair of distinct cells. By a slight abuse of notation, we will write in the rest of the paper $V_\rho(\mathbf{i})$ for the union of the sub-cubes $\bigcup_{\mathbf{j} \in V_\rho(\mathbf{i})} \mathbf{j}$.

Besides, the set of neighbors $V_\rho(\mathbf{i})$ can be re-written as $V_\rho(\mathbf{i}) = \{\mathbf{j} \in V_\rho, d(\mathbf{i}, \mathbf{j}) \leq R\}$. Hence $V_\rho(\mathbf{i})$ is a convex union of disjoint sub-cubes of volume $\lambda_d(\mathbf{W}_\rho)/N_\rho$, which are at most $(2R+1)^d$, and can be included in the cube $\mathfrak{C}_\rho$ defined in (1.7) up to a translation. Since f is translation invariant, we obtain

$$\mathbb{E}\left[\sum_{\substack{(C_1, C_2)_{\neq} \in \mathfrak{m}^2 \\ z(C_1), z(C_2) \in V_\rho(\mathbf{i})}} \mathbb{1}_{f(C_1) > v_\rho, f(C_2) > v_\rho} \right] \leq \frac{G_2(\rho)}{N_\rho}. \tag{2.12}$$

Using the fact that $\mathbb{P}(M_{f,\mathbf{i}} > v_\rho, M_{f,\mathbf{j}} > v_\rho | A_\rho) \leq \mathbb{P}(M_{f,\mathbf{i}} > v_\rho, M_{f,\mathbf{j}} > v_\rho) / \mathbb{P}(A_\rho)$ we deduce from (2.11) that

$$p_{\mathbf{ij}} := \mathbb{P}(M_{f,\mathbf{i}} > v_\rho, M_{f,\mathbf{j}} > v_\rho | A_\rho) \leq \frac{G_2(\rho)}{\mathbb{P}(A_\rho) N_\rho}. \tag{2.13}$$

Applying (2.6) to the conditional probability given A_ρ, we deduce from (2.8), (2.10), (2.13) and from the fact $|V_\rho| = N_\rho$ that

$$\left| \mathbb{P}(U'_{V_\rho}(v_\rho) \leq r - 1 | A_\rho) - e^{-\mu_\rho} \sum_{k=0}^{r-1} \frac{\mu_\rho^k}{k!} \right| \leq \frac{2(2R+1)^d}{\mathbb{P}(A_\rho)^2} \cdot \left(c(W)^2 \cdot \frac{(G_1(\rho) + \tau)^2}{N_\rho} + \mathbb{P}(A_\rho) G_2(\rho) \right).$$

The rate of convergence (2.4) results directly from the previous upper bound and the fact that $\mathbb{P}(A_\rho)$ and $G_1(\rho)$ converge respectively to 1 and 0 according to Conditions (TCP) and (FRC). □

Proof of Lemma 1. Let us notice that Lemma 1 is trivial when $r = 1$. More generally, for all $r \in \mathbf{N}^*$, we have

$$\left|\mathbb{P}\left(U_\rho(v_\rho) \leq r-1\right) - \mathbb{P}\left(U'_{V_\rho}(v_\rho) \leq r-1\right)\right| \leq 2\mathbb{P}\left(U_\rho(v_\rho) \neq U'_{V_\rho}(v_\rho)\right). \tag{2.14}$$

According to (1.3) and (2.1), the above random variables differ if and only if there are at least two exceedances in the same sub-cube $\mathbf{i}$, i.e.

$$\begin{aligned}\mathbb{P}\left(U_\rho(v_\rho) \neq U'_{V_\rho}(v_\rho)\right) &= \mathbb{P}\left(\bigcup_{\mathbf{i}\in V_\rho} \bigcup_{\substack{(C_1,C_2)_{\neq}\in\mathfrak{m}^2,\\ z(C_1),z(C_2)\in\mathbf{i}\cap\mathbf{W}_\rho}} \{f(C_1) > v_\rho, f(C_2) > v_\rho\}\right)\\ &\leq \sum_{\mathbf{i}\in V_\rho} \mathbb{E}\left[\sum_{\substack{(C_1,C_2)_{\neq}\in\mathfrak{m}^2\\ z(C_1),z(C_2)\in\mathbf{i}\cap\mathbf{W}_\rho}} \mathbb{1}_{f(C_1)>v_\rho, f(C_2)>v_\rho}\right].\end{aligned}$$

Since $|V_\rho| = N_\rho$, the right-hand side is bounded by $G_2(\rho)$ thanks to (2.12). This shows that $\mathbb{P}\left(U_\rho(v_\rho) \neq U'_{V_\rho}(v_\rho)\right) \leq G_2(\rho)$ and consequently we deduce (2.3) from (2.14). □

Proof of Lemma 3. From (2.2) and the triangle inequality, we get

$$|\mu_\rho - \tau| \leq |\mathbb{E}\left[U_\rho(v_\rho)\right] - \tau| + \mathbb{E}\left[U_\rho(v_\rho) - U'_{V_\rho}(v_\rho)\right],$$

where $U_\rho(v_\rho) \geq U'_{V_\rho}(v_\rho)$ a.s. According to (1.4) and (1.5), we obtain that

$$|\mu_\rho - \tau| \leq G_1(\rho) + \mathbb{E}\left[U_\rho(v_\rho) - U'_{V_\rho}(v_\rho)\right]. \tag{2.15}$$

To give an upper bound for the second term of the right-hand side of (2.15), we use the fact that the family V_ρ covers $\mathbf{W}_\rho$. Intuitively, the number of exceedance sub-cubes $U'_{V_\rho}(v_\rho)$ can be approximated by the number of exceedance cells $U_\rho(v_\rho)$ when $G_2(\rho)$ is negligible. We prove this fact below. From (1.3) and (2.1), we obtain a.s. that

$$\begin{aligned}U_\rho(v_\rho) - U'_{V_\rho}(v_\rho) &= \sum_{\mathbf{i}\in V_\rho} \sum_{\substack{C\in\mathfrak{m},\\ z(C)\in\mathbf{i}\cap\mathbf{W}_\rho}} \mathbb{1}_{f(C)>v_\rho} - \mathbb{1}_{M_{f,\mathbf{i}}>v_\rho}\\ &= \sum_{\mathbf{i}\in V_\rho} \left(\sum_{\substack{C\in\mathfrak{m},\\ z(C)\in\mathbf{i}\cap\mathbf{W}_\rho}} \mathbb{1}_{f(C)>v_\rho} - 1\right) \mathbb{1}_{M_{f,\mathbf{i}}>v_\rho}\\ &\leq \sum_{\mathbf{i}\in V_\rho} \sum_{\substack{(C_1,C_2)_{\neq}\in\mathfrak{m}^2\\ z(C_1),z(C_2)\in\mathbf{i}\cap\mathbf{W}_\rho}} \mathbb{1}_{f(C_1)>v_\rho, f(C_2)>v_\rho}.\end{aligned} \tag{2.16}$$

The last inequality comes from the fact that if there is 0 or 1 exceedance cell inside a sub-cube $\mathbf{i}$, the sums over $C \in \mathfrak{m}, z(C) \in \mathbf{i}\cap\mathbf{W}_\rho$ and $(C_1, C_2)_{\neq} \in \mathfrak{m}^2, z(C_1), z(C_2) \in \mathbf{i}\cap\mathbf{W}_\rho$ are null. Otherwise, if the number of exceedances is $k \geq 2$, we use the fact $k - 1 \leq \frac{k(k-1)}{2}$ which is the number of exceedance pairs.

Taking expectations in (2.16) and using the fact that the mean of the right-hand side of (2.16) is bounded by $G_2(\rho)$ as in the proof of Lemma 1, we get

$$\mathbb{E}\left[U_\rho(v_\rho) - U'_{V_\rho}(v_\rho)\right] \leq G_2(\rho). \tag{2.17}$$

It results from the previous inequality and (2.15) that $|\mu_\rho - \tau|$ is bounded by $G_1(\rho) + G_2(\rho)$. □

Remark 2. As mentioned in Section 1, p. 4, Theorem 1 can be extended to more general tessellations. Indeed, CONDITION (FRC) can be replaced by the following property: "$\beta(\rho) := N_\rho \sup_{\mathbf{i}\in V_\rho} \beta\left(\sigma(M_{f,\mathbf{i}}), \sigma(M_{f,\mathbf{j}}, d(\mathbf{i},\mathbf{j}) > R)\right)$ converges to 0 as ρ goes to infinity", where $\beta(\mathcal{A},\mathcal{B})$ is the β-mixing coefficient between two σ-algebras $\mathcal{A}$ and $\mathcal{B}$. This comes from the fact that b_3, which appears in (2.7), can be bounded by $\beta(\rho)$.

2.2 Proof of Theorem 2

By Kallenberg's theorem (see Proposition 3.22, p. 156 in [32], see also the proof of Theorem 2.1.2 in [10]) it is enough to check that:

- for all Borel subset $B \subset W$ and ${}_*x < s \leq t \leq x^*$

$$\mathbb{E}\left[\#\Phi_\rho \cap (B \times (s,t])\right] \underset{\rho\to\infty}{\longrightarrow} \mathbb{E}\left[\#\Phi \cap (B \times (s,t])\right], \tag{2.18}$$

- for all $\mathscr{P} = \bigcup_{l=1}^{L} B^{(l)} \times (s_l, t_l]$, where $B^{(l)}$ is the intersection of W and a rectangular solid in $\mathbf{C}^{(W)}$ and ${}_*x < s_l \leq t_l \leq x^*$

$$\mathbb{P}\left(\#\Phi_\rho \cap \mathscr{P} = 0\right) \underset{\rho\to\infty}{\longrightarrow} \mathbb{P}\left(\#\Phi \cap \mathscr{P} = 0\right). \tag{2.19}$$

Proof of (2.18)**.** From (1.1), we have

$$\mathbb{E}\left[\#\Phi_\rho \cap (B \times (s,t])\right] = \mathbb{E}\left[\sum_{\substack{C\in\mathbf{m},\\ z(C)\in\mathbf{B}_\rho}} \mathbb{1}_{a_\rho s + b_\rho < f(C) \leq a_\rho t + b_\rho}\right] = \lambda_d(\mathbf{B}_\rho)\cdot\left(\mathbb{P}\left(f(\mathcal{C}) > a_\rho s + b_\rho\right) - \mathbb{P}\left(f(\mathcal{C}) > a_\rho t + b_\rho\right)\right),$$

where $\mathbf{B}_\rho = \rho^{1/d}B$. Since $\lambda_d(\mathbf{B}_\rho) = \frac{\lambda_d(B)}{\lambda_d(W)}\cdot\lambda_d(\mathbf{W}_\rho)$ and $\lambda_d(\mathbf{W}_\rho)\cdot\mathbb{P}\left(f(\mathcal{C}) > v_\rho(t)\right)$ converges to $\tau(t)$ for all $t \in \mathbf{R}$, we get

$$\mathbb{E}\left[\#\Phi_\rho \cap (B \times (s,t])\right] \underset{\rho\to\infty}{\longrightarrow} \frac{\lambda_d(B)}{\lambda_d(W)}\cdot(\tau(s) - \tau(t)) = \mathbb{E}\left[\#\Phi \cap (B \times (s,t])\right] \tag{2.20}$$

and consequently we obtain (2.18). □

Proof of (2.19)**.** We can write $\mathscr{P}$ as a disjoint union of strips, i.e.

$$\mathscr{P} = \bigsqcup_{l=1}^{L} B^{(l)} \times J^{(l)} \tag{2.21}$$

such that the Borel subsets $B^{(l)} \subset W$ are disjoint and such that $J^{(l)}$ is a finite union of half-open intervals, $1 \leq l \leq L$. The following lemma shows that it is enough to investigate the case where $\mathscr{P}$ is a strip.

Lemma 4. *Let $\mathscr{P}$ be as in* (2.21)*. Under the hypothesis of Theorem 2, we have*

$$\mathbb{P}\left(\#\Phi_\rho \cap \mathscr{P} = 0\right) - \prod_{l=1}^{L} \mathbb{P}\left(\#\Phi_\rho \cap (B^{(l)} \times J^{(l)}) = 0\right) \underset{\rho\to\infty}{\longrightarrow} 0.$$

The proof of Lemma 4 is postponed at the end of the subsection. Thanks to Lemma 4, we can assume that $\mathscr{P}$, defined in (2.21), is only a strip $\mathscr{P} = B \times J$, where J is a finite union of half-open intervals and $B \subset W$. Without loss of generality, we can assume that these intervals are disjoint, i.e.

$$J = \bigsqcup_{j=1}^{k} (s_j, t_j] \tag{2.22}$$

with ${}_*x < s_j \leq t_j \leq x^*$ and $t_j \leq s_{j+1}$, $1 \leq j \leq k$. In the same spirit as in the proof of Theorem 1, we introduce the two random variables

$$\mathscr{U}_\rho(B \times J) = \#\Phi_\rho \cap (B \times J) = \sum_{\substack{C\in\mathfrak{m},\\ z(C)\in\mathbf{B}_\rho}} \mathbb{1}_{a_\rho^{-1}(f(C)-b_\rho)\in J} \text{ and } \mathscr{U}'_{V_\rho}(B \times J) = \sum_{\mathbf{i}\in V_\rho} \mathbb{1}_{a_\rho^{-1}(M_{f,\mathbf{i}}(B)-b_\rho)\in J}, \tag{2.23}$$

where

$$M_{f,\mathbf{i}}(B) = \max_{\substack{C\in\mathfrak{m},\\ z(C)\in\mathbf{i}\cap\mathbf{B}_\rho}} f(C).$$

In particular, $\mathscr{U}_\rho(W \times (s, \infty)) = U_\rho(v_\rho(s))$ and $\mathscr{U}'_{V_\rho}(W \times (s, \infty)) = U'_{V_\rho}(v_\rho(s))$, where $U_\rho(v_\rho(s))$ and $U'_{V_\rho}(v_\rho(s))$ are defined in (1.3) and (2.1). We denote by $\boldsymbol{\mu}_\rho(B \times J)$ the mean of $\mathscr{U}'_{V_\rho}(B \times J)$, i.e.

$$\boldsymbol{\mu}_\rho(B \times J) = \mathbb{E}\left[\mathscr{U}'_{V_\rho}(B \times J)\right] = \sum_{\mathbf{i}\in V_\rho} \mathbb{P}\left(a_\rho^{-1}(M_{f,\mathbf{i}}(B) - b_\rho) \in J\right).$$

As in the proof of Theorem 1, we subdivide the proof into three steps. More precisely, we show that

$$\mathbb{P}\left(\mathscr{U}_\rho(B \times J) = 0\right) - \mathbb{P}\left(\mathscr{U}'_{V_\rho}(B \times J) = 0\right) \underset{\rho\to\infty}{\longrightarrow} 0, \tag{2.24a}$$

$$\mathbb{P}\left(\mathscr{U}'_{V_\rho}(B \times J) = 0\right) - e^{-\boldsymbol{\mu}_\rho(B\times J)} \underset{\rho\to\infty}{\longrightarrow} 0, \tag{2.24b}$$

$$\boldsymbol{\mu}_\rho(B \times J) \underset{\rho\to\infty}{\longrightarrow} \nu(B \times J). \tag{2.24c}$$

Notice that the convergences (2.24a), (2.24b) and (2.24c) are generalisations of Lemmas 1, 2 and 3 respectively. For the proof of (2.24a), it is enough to show that $\mathbb{P}\left(\mathscr{U}_\rho(B \times J) \neq \mathscr{U}'_{V_\rho}(B \times J)\right)$ converges to 0 as ρ goes to infinity. Since $\mathscr{U}_\rho(B \times J) \geq \mathscr{U}'_{V_\rho}(B \times J)$ for all Borel subsets, we have

$$\begin{aligned}
\mathbb{P}\left(\mathscr{U}_\rho(B \times J) \neq \mathscr{U}'_{V_\rho}(B \times J)\right) &\leq \sum_{j=1}^{k} \mathbb{P}\left(\mathscr{U}_\rho(B \times (s_j, t_j]) \neq \mathscr{U}'_{V_\rho}(B \times (s_j, t_j])\right)\\
&\leq \sum_{j=1}^{k} \mathbb{P}\left(\mathscr{U}_\rho(W \times (s_j, \infty)) \neq \mathscr{U}'_{V_\rho}(W \times (s_j, \infty))\right)\\
&= \sum_{j=1}^{k} \mathbb{P}\left(U_\rho(v_\rho(s_j)) \neq U'_{V_\rho(v_\rho(s_j))}\right).
\end{aligned}$$

The proof of Lemma 1 shows that the right-hand side converges to 0.

Secondly, we prove (2.24b). In the same spirit as in the proof of Lemma 2, we apply Proposition 3 conditional on A_ρ to $X_{\mathbf{i}} = a_\rho^{-1}(M_{f,\mathbf{i}}(B) - b_\rho)$ and $J = \bigsqcup_{j=1}^k (s_j, t_j]$. Let $\mathbf{i} \in V_\rho$ and $\mathbf{j} \in V_\rho(\mathbf{i}) - \{\mathbf{i}\}$. Since $M_{f,\mathbf{i}}(B) \le M_{f,\mathbf{i}}$, we get

$$p_{\mathbf{i}} = \mathbb{P}\left(a_\rho^{-1}(M_{f,\mathbf{i}}(B) - b_\rho) \in J | A_\rho\right) \le \mathbb{P}\left(M_{f,\mathbf{i}}(B) > v_\rho(s_1) | A_\rho\right) = O(N_\rho^{-1})$$

according to (2.9). Moreover

$$\begin{aligned} p_{\mathbf{ij}} &= \mathbb{P}\left(a_\rho^{-1}(M_{f,\mathbf{i}}(B) - b_\rho) \in J, a_\rho^{-1}(M_{f,\mathbf{j}}(B) - b_\rho) \in J | A_\rho\right) \\ &\le \mathbb{P}\left(M_{f,\mathbf{i}}(B) > v_\rho(s_1), M_{f,\mathbf{i}}(B) > v_\rho(s_1) | A_\rho\right) = O\left(G_2(\rho) \cdot N_\rho^{-1}\right) \end{aligned}$$

according to (2.13). We deduce (2.24b) from the previous inequalities and Proposition 3.

Thirdly, we prove (2.24c). According to (2.22) and (2.23), we have a.s.

$$\mathscr{U}_\rho(B \times J) = \sum_{j=1}^k \#\Phi_\rho \cap (B \times (s_j, t_j]).$$

Taking expectations in the previous equality, we deduce from (2.20) that

$$\mathbb{E}\left[\mathscr{U}_\rho(B \times J)\right] \underset{\rho\to\infty}{\longrightarrow} \frac{\lambda_d(B)}{\lambda_d(W)} \sum_{j=1}^k (\tau(s_j) - \tau(t_j)) = \nu(B \times J). \tag{2.25}$$

Moreover

$$\mathbb{E}\left[\mathscr{U}_\rho(B \times J)\right] - \boldsymbol{\mu}_\rho(B \times J) = \mathbb{E}\left[\mathscr{U}_\rho(B \times J) - \mathscr{U}'_{V_\rho}(B \times J)\right] \le \mathbb{E}\left[U_\rho(v_\rho(s_1)) - U'_{V_\rho}(v_\rho(s_1))\right] \tag{2.26}$$

converges to 0 according to (2.17). We deduce (2.24c) from (2.25) and (2.26).

Finally, according to (2.24a), (2.24b), (2.24c) and the fact that $\mathscr{U}_\rho(B \times J) = \#\Phi_\rho \cap (B \times J)$, we deduce that

$$\mathbb{P}\left(\#\Phi_\rho \cap (B \times J) = 0\right) \underset{\rho\to\infty}{\longrightarrow} e^{-\nu(N \times J)} = \mathbb{P}\left(\#\Phi \cap (B \times J) = 0\right)$$

and consequently we obtain (2.19). □

The end of the subsection is devoted to the proof of Lemma 4.

Proof of Lemma 4. Let $\mathscr{P} = \bigsqcup_{l=1}^L B^{(l)} \times J^{(l)}$ and $B^{(l)} = B_l \cap W$ such that the rectangular solids $B_l \subset \mathbf{C}^{(W)}$ are disjoint. We denote respectively by $V_\rho(B^{(l)})$, $S_\rho(B^{(l)})$ and $V_\rho^\circ(B^{(l)})$ the sets

$$\begin{cases} V_\rho(B^{(l)}) = \{\mathbf{i} \in V_\rho, \mathbf{i} \cap B_l \neq \varnothing\}, \\ S_\rho(B^{(l)}) = \{\mathbf{i} \in V_\rho, \mathbf{i} \cap \partial B_l \neq \varnothing\}, \\ V_\rho^\circ(B^{(l)}) = \{\mathbf{i} \in V_\rho(B^{(l)}), d(\mathbf{i}, S_\rho(B^{(l)})) > R\}. \end{cases}$$

Finally, we consider the random variable $\mathscr{U}'_{V_\rho^\circ}(B^{(l)} \times J^{(l)}) \le \mathscr{U}_\rho(B^{(l)} \times J^{(l)})$ defined as

$$\mathscr{U}'_{V_\rho^\circ}(B^{(l)} \times J^{(l)}) = \sum_{\mathbf{i} \in V_\rho^\circ(B^{(l)})} \mathbb{1}_{a_\rho^{-1}(M_{f,\mathbf{i}}(B^{(l)}) - b_\rho) \in J^{(l)}}.$$

Let $1 \leq l \leq L$ be fixed. Since B_l is a rectangular solid in $\mathbf{C}^{(W)}$ which is covered with at most N_ρ sub-cubes $\mathbf{i}$, we have $\#S_\rho(B^{(l)}) \leq c \cdot N_\rho^{(d-1)/d}$. This shows that

$$\mathbb{P}\left(\mathscr{U}'_{V_\rho^\circ}(B^{(l)} \times J^{(l)}) \neq \mathscr{U}'_{V_\rho}(B^{(l)} \times J^{(l)})\right) \leq \#S_\rho(B^{(l)}) \cdot \mathbb{P}\left(M_{f,\mathbf{i}} > v_\rho\right) = O\left(N_\rho^{-1/d}\right)$$

according to (2.9) and Condition (TCP). Thanks to (2.24a), we deduce that

$$\mathbb{P}\left(\mathscr{U}_\rho(B^{(l)} \times J^{(l)}) = 0\right) - \mathbb{P}\left(\mathscr{U}'_{V_\rho^\circ}(B^{(l)} \times J^{(l)}) = 0\right) \underset{\rho\to\infty}{\longrightarrow} 0. \tag{2.27}$$

Moreover, conditional on A_ρ, the random variables $\mathscr{U}'_{V_\rho^\circ}(B^{(l)} \times J^{(l)})$ are independent since $\mathscr{U}'_{V_\rho^\circ}(B^{(l)} \times J^{(l)})$ is $\sigma\left(M_{f,\mathbf{i}}, \mathbf{i} \in V_\rho^\circ(B^{(l)})\right)$ measurable and $d\left(V_\rho^\circ(B^{(l)}), V_\rho^\circ(B^{(l')})\right) > R$ for all $1 \leq l \neq l' \leq L$. In particular, we get

$$\mathbb{P}\left(\bigcap_{l=1}^{L}\left\{\mathscr{U}'_{V_\rho^\circ}(B^{(l)} \times J^{(l)}) = 0\right\}\Big| A_\rho\right) = \prod_{l=1}^{L}\mathbb{P}\left(\mathscr{U}'_{V_\rho^\circ}(B^{(l)} \times J^{(l)}) = 0\Big| A_\rho\right).$$

Lemma 4 is a consequence of the previous equality, the convergence (2.27) and the fact that

$$\mathbb{P}\left(\#\Phi_\rho \cap \mathscr{P} = 0\right) = \mathbb{P}\left(\bigcap_{l=1}^{L}\{\mathscr{U}_\rho(B^{(l)} \times J^{(l)}) = 0\}\right).$$

□

Remark 3. The inequalities appearing in (1.5), (1.6) and Theorem 1 have to be reversed when we deal with the r smallest values. This fact will be extensively used in the rest of the paper.

In the three following sections, we apply Theorem 1 to derive the asymptotic behaviours of the order statistics for different geometrical characteristics and random tessellations. For aesthetic reasons, we only investigate maxima and minima for the particular case $W = \mathbf{C}^{(W)} = [0,1]^d$ keeping in mind that these results can be generalized to order statistics and to any bounded set with $\lambda_d(W) \neq 0$. Up to a normalization, all the thresholds v_ρ can be written as $v_\rho = v_\rho(t) = a_\rho t + b_\rho$ (excepted in Section 5) so that Theorem 2 is also available.

3 Extreme Values of a Poisson-Delaunay tessellation

Before applying Theorem 1 to different geometrical characteristics of a Poisson-Delaunay tessellation, we introduce some notation and preliminaries.

Notation. For all $z \in \mathbf{R}^d$ and $r \geq 0$, we denote by $B(z,r)$ and $S(z,r)$ the ball and the sphere of radius r centered in z. When $z = 0$ and $r = 1$, we denote by $\mathbf{S}^{d-1} = S(0,1)$ the unit sphere and $\kappa_d = \lambda_d(B(0,1))$. Let C be a simplex in $\mathbf{R}^d$, i.e. the convex hull of $(d+1)$ points. When the extremal points of C are affinely independent, we denote by $B(C)$, $S(C)$, $z(C)$ and $R(C)$ the circumball, the circumsphere, the circumcenter and the circumradius of C respectively. Otherwise, we take $B(C) = S(C) = \{0\}$, $z(C) = 0$ and $R(C) = 0$.

Let k be an integer and $x_1, \ldots, x_k$ be k points in $\mathbf{R}^d$ and let f be a function defined on $\mathcal{K}_d$ with values on some measurable space.

- For all $r \geq 0$, we denote by $r\mathbf{x}_{1:k}$ the k-tuple $(rx_1, \ldots, rx_k)$ and by $\{r\mathbf{x}_{1:k}\}$ the set of points $\{rx_1, \ldots, rx_k\}$.
- When $k = d+1$, we denote by $\Delta(\mathbf{x}_{1:d+1})$ the convex hull of $x_1, \ldots, x_{d+1}$ and $f(\mathbf{x}_{1:d+1}) = f\left(\Delta(\mathbf{x}_{1:d+1})\right)$.

- If $k \leq d+1$ and if $\{\mathbf{y}_{k+1:d+1}\} = \{y_{k+1}, \ldots, y_{d+1}\}$ is a set of $d+1-k$ points in $\mathbf{R}^d$, we denote by $\Delta(\mathbf{x}_{1:k}, \mathbf{y}_{k+1:d+1})$ the convex hull of $x_1, \ldots, x_k, y_{k+1}, \ldots, y_{d+1}$ and $f(\mathbf{x}_{1:k}, \mathbf{y}_{k+1:d+1}) = f\left(\Delta\left(\mathbf{x}_{1:k}, \mathbf{y}_{k+1:d+1}\right)\right)$. In particular, if $k = 0$, we take $f(\mathbf{x}_{1:0}, \mathbf{y}_{1:d+1}) = f(\mathbf{y}_{1:d+1})$.

- Finally, we denote by $d\sigma(u)$ and $d\sigma(\mathbf{u}_{1:d+1}) = d\sigma(u_1) \cdots d\sigma(u_{d+1})$ the uniform distribution over $\mathbf{S}^{d-1}$ and the product measure respectively.

Preliminaries. Let χ be a locally finite subset of $\mathbf{R}^d$ such that each subset of size $n < d+1$ are affinely independent and no $d+2$ points lie on a sphere. If $d+1$ points $x_1, \ldots, x_{d+1}$ of χ lie on a ball that contains no point of χ in its interior, then the convex hull of $x_1, \ldots, x_{d+1}$ is called a cell. The set of such cells defines a partition of $\mathbf{R}^d$ into simplices and such partition is called the Delaunay tessellation (see [35], p. 478). Such model is the key ingredient of the first algorithm for computing the minimum spanning tree [37]. It is extensively used in medical image segmentation [39], in finite element method to build meshes [14] and is a powerful tool for reconstructing a $3D$ set from a discrete point set [34].

When $\chi = \mathbf{X}$ is a Poisson point process, we speak about Poisson-Delaunay tessellation denoted by $\mathfrak{m}_{PDT}$. For each cell $C \in \mathfrak{m}_{PDT}$ which is a.s. a simplex, the nucleus $z(C)$ is defined as the circumcenter of C. The relation between the intensity γ of $\mathfrak{m}_{PDT}$ and the intensity $\gamma_{\mathbf{X}}$ of the underlying Poisson point process is given by (see Section 7 in [22]) $\gamma = \beta_d^{-1} \cdot \gamma_{\mathbf{X}}$, where

$$\beta_d = \frac{(d^3 + d^2)\Gamma\left(\frac{d^2}{2}\right)\Gamma^d\left(\frac{d+1}{2}\right)}{\Gamma\left(\frac{d^2+1}{2}\right)\Gamma^d\left(\frac{d+2}{2}\right)2^{d+1}\pi^{\frac{d-1}{2}}}. \tag{3.1}$$

In particular, we have $\beta_1 = 1$, $\beta_2 = \frac{1}{2}$ and $\beta_3 = \frac{35}{24\pi^2}$. To be in the framework of Theorem 1, we assume without loss of generality that $\gamma = 1$, i.e. $\gamma_{\mathbf{X}} = \beta_d$. Moreover, we partition the window $\mathbf{W}_\rho = \rho^{1/d}[0,1]^d$ into $N_\rho := \left\lfloor \left(\frac{\rho}{2\log\rho}\right)^{1/d} \right\rfloor^d$ sub-cubes $\mathbf{i} \in V_\rho$, where $\lfloor x \rfloor$ denotes the integer part of x for any $x \in \mathbf{R}$. We think that this choice for N_ρ is quasi-optimal for the rate of convergence in Theorem 1. Indeed, according to (1.8), this rate is of order $G_1(\rho) + G_2(\rho) + N_\rho^{-1} + \mathbb{P}\left(A_\rho^c\right)$, where $G_1(\rho)$ is independent of N_ρ. Besides, according to (3.4), we have $\mathbb{P}\left(A_\rho^c\right) \leq N_\rho e^{-\rho/N_\rho}$. This implies that $N_\rho e^{-\rho/N_\rho}$ has to converge to 0 so that $N_\rho = O\left(\frac{\rho}{\log\rho}\right)$. Moreover, the larger N_ρ is, the smaller N_ρ^{-1} and $G_2(\rho)$ are. If $N_\rho \underset{\rho\to\infty}{\sim} \frac{a\cdot\rho}{\log\rho}$ for some constant $a < 1$, the choice of a does not affect the order of $G_2(\rho)$. Hence, N_ρ seems to be optimal when it minimizes $N_\rho^{-1} + N_\rho e^{-\rho/N_\rho}$ i.e. when N_ρ is close to $\frac{\rho}{2\log\rho}$.

To apply Theorem 1, we first check Condition (FRC) for any measurable function $f : \mathcal{K}_d \to \mathbf{R}$. To do it, we define the event A_ρ as

$$A_\rho = \bigcap_{\mathbf{i}\in V_\rho} \{\mathbf{X} \cap \mathbf{i} \neq \varnothing\}. \tag{3.2}$$

Lemma 5. *Let $f : \mathcal{K}_d \to \mathbf{R}$ be a measurable function. Then* Condition (FRC) *is satisfied for $R = 2 \cdot \left(\lfloor\sqrt{d}\rfloor + 1\right)$ and for the event A_ρ defined in* (3.2).

Proof of Lemma 5. We use the same arguments as in the proof of Proposition 3 in [3]. Let $\mathbf{i} \in V_\rho$ be a sub-cube in $\mathbf{W}_\rho$ and let $C \in \mathfrak{m}_{PDT}$ such that $z(C) \in \mathbf{i}$. Since a $(d+1)$-tuple of points of $\mathbf{X}$ is a Delaunay cell if and only if its circumball contains no point in its interior, we have $R(C) = \min_{x\in\mathbf{X}}\{|z(C) - x|\}$. Moreover, conditional on A_ρ, there exists a point x_0 in $\mathbf{X} \cap \mathbf{i}$. In particular, we have $|z(C) - x_0| \leq \sqrt{d} \cdot c_\rho$, where c_ρ is the length of the sides of each sub-cube. Consequently, we obtain

$$R(C) \leq \sqrt{d} \cdot c_\rho. \tag{3.3}$$

This shows that the circumsphere $S(C)$ of C is included in $V_\rho(\mathbf{i}, D) := \{\mathbf{j} \in V_\rho, d(\mathbf{i}, \mathbf{j}) \leq D\}$, where $D = \lfloor\sqrt{d}\rfloor + 1$. Indeed if not, there exists a point $y \in S(C)$ such that y is in a sub-cube $\mathbf{j}$ with $d(\mathbf{i}, \mathbf{j}) \geq D + 1$. This shows that $|y - z(C)| > (\lfloor\sqrt{d}\rfloor + 1) \cdot c_\rho$ and contradicts (3.3) since $R(C) = |y - z(C)|$.

Since $S(C)$ is included in $V_\rho(\mathbf{i}, D)$ for any cell $C \in \mathfrak{m}_{PDT}$ such that $z(C) \in \mathbf{i}$, this shows that $M_{f,\mathbf{i}}$ is $\sigma(\mathbf{X} \cap V_\rho(\mathbf{i}, D))$ measurable. Because $d(A, B) > 2D$ implies that $\{\mathbf{i}, d(\mathbf{i}, A) < D\}$ and $\{\mathbf{i}, d(\mathbf{i}, B) < D\}$ are disjoint and because $\mathbf{X} \cap \{\mathbf{i}, d(\mathbf{i}, A) < D\}$ and $\mathbf{X} \cap \{\mathbf{i}, d(\mathbf{i}, B) < D\}$ are independent, the σ-algebras $\sigma(M_{f,\mathbf{i}}, \mathbf{i} \in A)$ and $\sigma(M_{f,\mathbf{i}}, \mathbf{i} \in B)$ are independent, yielding $R = 2D = 2 \cdot \left(\lfloor\sqrt{d}\rfloor + 1\right)$.

Moreover the probability of the event A_ρ converges to 1. Indeed, since $\mathbf{X}$ is a Poisson point process, we get

$$\mathbb{P}(A_\rho^c) = \mathbb{P}\left(\bigcup_{\mathbf{i} \in V_\rho} \{\mathbf{X} \cap \mathbf{i} = \varnothing\}\right) \leq N_\rho e^{-\rho/N_\rho} = O\left((\log\rho)^{-1} \times \rho^{-1}\right). \tag{3.4}$$

□

Besides, the distribution function of the typical cell can be made explicit. Indeed, let $f : \mathcal{K}_d \to \mathbf{R}$ be a translation invariant function on the set of convex bodies. An integral representation of $f(\mathcal{C})$, due to Miles [20] (the proof can also be found in Theorem 10.4.4. of [35]), is given by

$$\mathbb{E}\left[f(\mathcal{C})\right] = \delta'_d \cdot \int_0^\infty \int_{(\mathbf{S}^{d-1})^{d+1}} r^{d^2-1} e^{-\delta_d r^d} \lambda_d(\mathbf{u}_{1:d+1}) f(r\mathbf{u}_{1:d+1}) dr d\sigma(\mathbf{u}_{1:d+1}) \text{ with } \delta'_d = (d+1)\beta_d \text{ and } \delta_d = \kappa_d \beta_d. \tag{3.5}$$

For practical reasons, we write below a generic lemma which gives an integral representation of the function $G_2(\cdot)$ defined in (1.6). To do it, we introduce some notation. As defined in (1.6), $G_2(\cdot)$ involves two different simplices Δ_1, Δ_2 such that $f(\Delta_i) > v_\rho$ and $z(\Delta_i) \in \mathfrak{C}_\rho$, $i = 1, 2$. The intersection of these cells is a k-dimensional simplex with $0 \leq k \leq d - 1$. Translating the circumcenter of the cell which has the largest circumradius say Δ_1 at the origin, the cells can be written as $\Delta_1 = \Delta(r\mathbf{u}_{1:d+1})$ and $\Delta_2 = \Delta(r\mathbf{u}_{1:k}, \mathbf{y}_{k+1:d+1})$ with $r \geq 0$, $u_1, \ldots, u_{d+1} \in \mathbf{S}^{d-1}$ and $y_{k+1}, \ldots, y_{d+1} \in \mathbf{R}^d$. We consider the two properties:

$$\mathscr{P}_1 : f(r\mathbf{u}_{1:k}, \mathbf{y}_{k+1:d+1}) > v_\rho, R(r\mathbf{u}_{1:k}, \mathbf{y}_{k+1:d+1}) \leq r \text{ and } z(r\mathbf{u}_{1:k}, \mathbf{y}_{k+1:d+1}) \in \mathfrak{C}_\rho, \tag{3.6a}$$

$$\mathscr{P}_2 : y_j \notin B(r\mathbf{u}_{1:d+1}) \text{ and } ru_j \notin B(r\mathbf{u}_{1:k}, \mathbf{y}_{k+1:d+1}) \text{ for all } k+1 \leq j \leq d+1. \tag{3.6b}$$

The first property concerns the cell Δ_2 which has the smallest circumradius whereas the second property means that the two simplices are Delaunay cells. Moreover, we introduce the set

$$E_{k,r,\mathbf{u}_{1:d+1}} = \{\mathbf{y}_{k+1:d+1} \in (\mathbf{R}^d)^{d+1-k} \text{ satisfying } \mathscr{P}_1 \text{ and } \mathscr{P}_2\}. \tag{3.7}$$

Finally, in the same spirit as in (3.5), we consider the volume of the union of the two circumballs, i.e.

$$\lambda_d^{(\cup)}(r, \mathbf{u}_{1:k}, \mathbf{y}_{k+1:d+1}) = \lambda_d\left(B(0, r) \cup B(r\mathbf{u}_{1:k}, \mathbf{y}_{k+1:d+1})\right). \tag{3.8}$$

We are now prepared to state the generic lemma.

Lemma 6. *Let $\mathfrak{m}_{PDT}$ be a Poisson-Delaunay tessellation of intensity $\gamma = 1$. Then*

$$G_2(\rho) = 2 \cdot \sum_{k=0}^{d} G_{2,k}(\rho), \tag{3.9}$$

where

$$G_{2,k}(\rho) = \rho \int_0^\infty \int_{(\mathbf{S}^{d-1})^{d+1}} \int_{(\mathbf{R}^d)^{d+1-k}} g_{2,k}(\rho, r, \mathbf{u}_{1:d+1}, \mathbf{y}_{k+1:d+1}) d\mathbf{y}_{k+1:d+1} d\sigma(\mathbf{u}_{1:d+1}) dr \tag{3.10}$$

and

$$g_{2,k}(\rho, r, \mathbf{u}_{1:d+1}, \mathbf{y}_{k+1:d+1}) = r^{d^2-1} e^{-\beta_d \lambda_d^{(\cup)}(r, \mathbf{u}_{1:k}, \mathbf{y}_{k+1:d+1})} \lambda_d(\mathbf{u}_{1:d+1}) \mathbb{1}_{f(r\mathbf{u}_{1:d+1})>v_\rho} \mathbb{1}_{E_{k,r,\mathbf{u}_{1:d+1}}}(\mathbf{y}_{k+1:d+1}). \quad (3.11)$$

Proof of Lemma 6. This will be sketched since it is in the same spirit as in the proof of (3.5). Considering that the intersection of the two Delaunay cells Δ_1, Δ_2 which appear in (1.6) is a k-dimensional simplex with $0 \leq k \leq d$ and assuming that $R(\Delta_1) \geq R(\Delta_2)$, we have

$$G_2(\rho) = 2 \sum_{k=0}^{d} \mathbb{E}\Bigg[\sum_{\substack{(x_1,\ldots,x_{d+1})_{\neq} \in \mathbf{X}^{d+1} \\ (y_1,\ldots,y_k)_{\neq} \in \mathbf{X}^k}} \mathbb{1}_{f(\mathbf{x}_{d+1})>v_\rho} \mathbb{1}_{f(\mathbf{x}_{1:k}, \mathbf{y}_{k+1:d+1})>v_\rho} \mathbb{1}_{R(\mathbf{x}_{1:d+1}) \geq R(\mathbf{x}_{1:k}, \mathbf{y}_{k+1:d+1})}$$
$$\times \mathbb{1}_{\mathbf{X} \cap B^{(\cup)}(\mathbf{x}_{1:d+1}, \mathbf{y}_{k+1:d+1}) \setminus \{\mathbf{x}_{1:d+1}\} \cup \{\mathbf{y}_{k+1:d+1}\}} = \varnothing \Bigg],$$

where $B^{(\cup)}(\mathbf{x}_{1:d+1}, \mathbf{y}_{k+1:d+1}) = B(\mathbf{x}_{1:d+1}) \cup B(\mathbf{x}_{1:k}, \mathbf{y}_{k+1:d+1})$. It results from Slivnyak-Mecke formula (see e.g. Theorem 3.3.5 in [35]) that

$$G_2(\rho) = 2 \sum_{k=0}^{d} \int_{(\mathbf{R}^d)^{d+1}} \int_{(\mathbf{R}^d)^{d+1-k}} \mathbb{1}_{f(\mathbf{x}_{d+1})>v_\rho} \mathbb{1}_{f(\mathbf{x}_{1:k}, \mathbf{y}_{k+1:d+1})>v_\rho} \mathbb{1}_{R(\mathbf{x}_{1:d+1}) \geq R(\mathbf{x}_{1:k}, \mathbf{y}_{k+1:d+1})}$$
$$\times \mathbb{P}\left(\#\mathbf{X} \cap B^{(\cup)}(\mathbf{x}_{1:d+1}, \mathbf{y}_{k+1:d+1}) = 0 \right) d\mathbf{y}_{k+1:d+1} d\mathbf{x}_{1:d+1}.$$

We conclude the proof of Lemma 6 by noting that $\#\mathbf{X} \cap B^{(\cup)}(\mathbf{x}_{1:d+1}, \mathbf{y}_{k+1:d+1})$ is Poisson distributed with mean $\beta_d \lambda_d \left(B^{(\cup)}(\mathbf{x}_{1:d+1}, \mathbf{y}_{k+1:d+1}) \right)$ and using for all $y_{k+1}, \ldots, y_{d+1}$ the (Blaschke-Petkantschin type) change of variables

$$\begin{aligned} \phi_1 : & \mathbf{R}_+ \times \mathbf{R}^d \times (\mathbf{S}^{d-1})^{d+1} \longrightarrow (\mathbf{R}^d)^{d+1} \\ & (r, z, \mathbf{u}_{1:d+1}) \longmapsto \mathbf{x}_{1:d+1} \text{ with } x_i = z + r u_i, \end{aligned} \quad (3.12)$$

where the Jacobian is $|D\phi_1(r, z, \mathbf{u}_{1:d+1})| = r^{d^2-1} \lambda_d(\mathbf{u}_{1:d+1})$.

□

In (3.6a), we have assumed that $R(r\mathbf{u}_{1:k}, \mathbf{y}_{k+1:d+1})$ is less than r. This property will be needed in Sections 3.2 and 3.3 to bound $\lambda_{d+1-k}(E_{k,r,\mathbf{u}_{1:d+1}})$ with respect to r (see Lemmas 8 and 9) but not in Section 3.1 since we could bound $\lambda_{d+1-k}(E_{k,r,\mathbf{u}_{1:d+1}})$ there with respect to v_ρ (as defined in (3.15)) instead of r.

3.1 Minimum of the circumradii

In this subsection, we investigate the minimum of circumradii, i.e.

$$R_{\min,PDT}(\rho) = \min_{\substack{C \in \mathfrak{m}_{PDT}, \\ z(C) \in \mathbf{W}_\rho}} R(C).$$

The asymptotic behaviour of $R_{\min,PDT}(\rho)$ is given in the following proposition.

Proposition 4. *Let $\mathfrak{m}_{PDT}$ be a Poisson-Delaunay tessellation of intensity $\gamma = 1$ in $\mathbf{R}^d$, $d \geq 2$. Then for all $t \geq 0$*

$$\left| \mathbb{P}\left(\alpha_{d,1}^{1/d} \rho^{1/d} R_{\min,PDT}(\rho)^d \geq t \right) - e^{-t^d} \right| = O\left(\rho^{-1/d} \right), \quad (3.13)$$

where

$$\alpha_{d,1} = \frac{\delta_d^d}{d!} = \frac{(\kappa_d \beta_d)^d}{d!} = \frac{1}{d!} \cdot \left(\frac{(d^3+d^2)\Gamma\left(\frac{d^2}{2}\right)\Gamma^d\left(\frac{d+1}{2}\right)\pi^{1/2}}{2^{d+1}\Gamma\left(\frac{d^2+1}{2}\right)\Gamma^{d+1}\left(\frac{d+2}{2}\right)} \right)^d .$$

Notice that the asymptotic behaviour of the maximum of circumradii has been investigated in [8].

Proof of Proposition 4. First, we give the asymptotic behaviour of the distribution function of $R(\mathcal{C})$. According to (3.5), the random variable $R(\mathcal{C})^d$ is Gamma distributed with parameters $\left(d^2, \delta_d^{-1}\right)$. Thanks to consecutive integrations by parts, this proves that

$$\mathbb{P}(R(\mathcal{C}) < v) = \sum_{i=d}^{\infty} \frac{1}{i!} (\delta_d v^d)^i e^{-\delta_d v^d} \tag{3.14}$$

for all $v \geq 0$. A Taylor approximation of the right-hand side when v goes to 0 shows that $|\mathbb{P}(R(\mathcal{C}) < v) - \alpha_{d,1} \cdot v^{d^2}|$ is of order v^{d^2+d}. Hence, taking for all $t \geq 0$

$$v_\rho = v_\rho(t) = \left(\alpha_{d,1}^{-1} \rho^{-1}\right)^{1/d^2} t^{1/d}, \tag{3.15}$$

we obtain

$$G_1(\rho) = |\rho \mathbb{P}(R(\mathcal{C}) < v_\rho) - t^d| = O\left(\rho^{-1/d}\right). \tag{3.16}$$

To calculate an upper bound of $G_2(\rho)$, it is enough to give a suitable upper bound for $G_{2,k}(\rho)$, $0 \leq k \leq d$ according to Lemma 6. Bounding the exponential in (3.11) by 1 (a suitable estimate when considering small cells) and $\lambda_d(\mathbf{u}_{1:d+1})$ by a constant, we deduce for all $r \in \mathbf{R}_+$, $\mathbf{u}_{1:d+1} \in (\mathbf{S}^{d-1})^{d+1}$ and $\mathbf{y}_{k+1:d+1} \in (\mathbf{R}^d)^{d+1-k}$ that

$$g_{2,k}(\rho, r, \mathbf{u}_{1:d+1}, \mathbf{y}_{k+1:d+1}) \leq c \cdot r^{d^2-1} \mathbb{1}_{r < v_\rho} \mathbb{1}_{E_{k,r,\mathbf{u}_{1:d+1}}}(\mathbf{y}_{k+1:d+1}). \tag{3.17}$$

When $k = 0$, we bound $\mathbb{1}_{E_{0,r,\mathbf{u}_{1:d+1}}}(\mathbf{y}_{1:d+1})$ by $\mathbb{1}_{R(\mathbf{y}_{1:d+1}) < r} \cdot \mathbb{1}_{z(\mathbf{y}_{1:d+1}) \in \mathfrak{C}_\rho}$. Integrating the right-hand side of (3.17) and taking the same change of variables as in (3.12), i.e. $y_i = z' + r' u_i'$, $1 \leq i \leq d+1$, we deduce from (3.10) and (3.15) that

$$G_{2,0}(\rho) \leq c \cdot \rho \lambda_d(\mathfrak{C}_\rho) \int_0^{v_\rho} r^{d^2-1} \int_0^r r'^{d^2-1} dr' dr = O\left(\log \rho \cdot \rho^{-1}\right). \tag{3.18}$$

When $1 \leq k \leq d$, we use the fact that $R(r\mathbf{u}_{1:k}, \mathbf{y}_{k+1:d+1}) < r$ implies $y_i \in B(ru_1, 2r)$ for all $k+1 \leq i \leq d+1$. Bounding $\mathbb{1}_{E_{k,r,\mathbf{u}_{1:d+1}}}(\mathbf{y}_{k+1:d+1})$ by $\mathbb{1}_{y_{k+1},\ldots,y_{d+1} \in B(ru_1,2r)}$ and integrating (3.17), we deduce from (3.10) that

$$\begin{aligned} G_{2,k}(\rho) &\leq c \cdot \rho \int_0^{v_\rho} \int_{\mathbf{S}^{d-1}} \int_{(\mathbf{R}^d)^{d+1}} r^{d^2-1} \mathbb{1}_{y_{k+1},\ldots,y_{d+1} \in B(ru_1,2r)} d\mathbf{y}_{k+1:d+1} d\sigma(u_1) dr \\ &\leq c \cdot \rho \int_0^{v_\rho} r^{d^2-1} \times r^{d(d+1-k)} dr = O\left(\rho^{-(d+1-k)/d}\right). \end{aligned} \tag{3.19}$$

Since $1 \leq k \leq d$, the right-hand side of (3.19) is less than $\rho^{-1/d}$ for ρ large enough. Indeed, $G_{2,k}(\rho)$ is maximal when $k = d$, i.e. when the two distinct Delaunay cells have d common vertices. From (3.9), (3.18) and (3.19) we deduce that $G_2(\rho) = O\left(\rho^{-1/d}\right)$ since $d \geq 2$. This together with (3.16) and Theorem 1 concludes the proof of Proposition 4. □

When $d=1$, the same method shows that $2\rho R_{\min,PDT}(\rho)$ converges to a standard exponential distribution. The only difference with the case $d \geq 2$ is that the rate of convergence is $\log \rho \cdot \rho^{-1}$ (and not ρ^{-1}) according to (3.18) and (3.19).

Let us remark that a slightly weaker version of Proposition 4 in $\mathbf{R}^d$ could have been deduced from a theorem due to Schulte and Thäle (see Theorem 1.1 in [36]). It comes from the fact that $R_{\min,PDT}(\rho)$ can be written as a minimum of a U-statistic. More precisely

$$R_{\min,PDT}(\rho) = \min_{\substack{\mathbf{x}_{1:d+1} \in \mathbf{X}^{d+1}, \\ z(\mathbf{x}_{1:d+1}) \in \mathbf{W}_\rho}} R(\mathbf{x}_{1:d+1}).$$

Indeed, if a simplex induced by a set of $(d+1)$ distinct points $\mathbf{x}_{1:d+1}$ of $\mathbf{X}$ minimizes the circumradius, it is necessarily a Delaunay cell: otherwise, the circumball $B(\mathbf{x}_{1:d+1})$ contains a point of $\mathbf{X}$ in its interior which contradicts the minimality of $R(\mathbf{x}_{1:d+1})$. Nevertheless, the rate of convergence $O\left(\rho^{-1/d}\right)$ of Proposition 4 is more accurate than the rate deduced from Theorem 1.1. in [36] since the latter is of order $O\left(\rho^{-1/2d}\right)$. To the best of our knowledge, the convergence of the point process provided by Theorem 2 applied to the circumscribed radius of Delaunay cells is new.

3.2 Maximum of the areas, $d = 2$

Here and in the subsequent subsection, we investigate the extremes of the areas of a planar Poisson-Delaunay tessellation of intensity 1. The extension to higher dimension would be intricate since the integral formula for the distribution function of the volume of the typical cell becomes intractable. The intensity of the underlying Poisson point process is

$$\gamma_{\mathbf{X}} = \beta_2 = \frac{1}{2}. \tag{3.20}$$

In this subsection, we investigate the maximum of the areas, i.e.

$$A_{\max,PDT}(\rho) = \max_{\substack{C \in \mathfrak{m}_{PDT}, \\ z(C) \in \mathbf{W}_\rho}} \lambda_2(C).$$

The following proposition shows that $A_{\max,PDT}(\rho)$ is of order $\log \rho$.

Proposition 5. *Let $\mathfrak{m}_{PDT}$ be a Poisson-Delaunay tessellation of intensity $\gamma = 1$ in $\mathbf{R}^2$. Then for all $t \in \mathbf{R}$*

$$\left| \mathbb{P}\left(\alpha_2 A_{\max,PDT}(\rho) - \log\left(\frac{3}{2}\rho\right) \leq t \right) - e^{-e^{-t}} \right| = O\left(1/\log \rho\right), \tag{3.21}$$

where $\alpha_2 = \frac{2\pi}{3\sqrt{3}}$.

Proof of Proposition 5. Thanks to (3.5), the distribution function of $\lambda_2(\mathcal{C})$ can be made explicit. Indeed, when $d=2$, an integral representation of $\mathbb{P}\left(\lambda_2(\mathcal{C}) > v\right)$ due to Rathie (see (3.2) in [31]) is

$$\mathbb{P}(\lambda_2(\mathcal{C}) > v) = \frac{6}{\pi} \int_{\alpha_2 \beta_2 v}^{\infty} x K_{1/6}^2(x) dx, \tag{3.22}$$

where $K_{1/6}(\cdot)$ denotes the modified Bessel function of order 1/6. When x goes to infinity, a Taylor approximation of $K_{1/6}(x)$ is given by (see Formula 9.7.2, p. 378 in [1])

$$K_{1/6}(x) = \sqrt{\frac{\pi}{2x}} e^{-x} \left(1 + O\left(\frac{1}{x}\right)\right). \tag{3.23}$$

We deduce from (3.20), (3.22) and (3.23) that for v large enough

$$\left|\mathbb{P}\left(\lambda_2(\mathcal{C}) > v\right) - \frac{3}{2}e^{-\alpha_2 v}\right| \leq c \cdot \int_{\frac{1}{2}\alpha_2 v}^{\infty} \frac{e^{-2x}}{x} dx \leq c \cdot \frac{e^{-\alpha_2 v}}{v}. \tag{3.24}$$

Taking for all $t \in \mathbf{R}$

$$v_\rho = v_\rho(t) = \frac{1}{\alpha_2}\left(\log\left(\frac{3}{2}\rho\right) + t\right), \tag{3.25}$$

we obtain from (3.24) that

$$G_1(\rho) = \left|\rho\mathbb{P}\left(\lambda_2(\mathcal{C}) > v_\rho\right) - e^{-t}\right| = O\left(1/\log\rho\right).$$

In the rest of the proof, we give a suitable upper bound for $G_2(\rho)$. Taking $f(\cdot) = \lambda_2(\cdot)$ in (3.11) and using the fact that $\lambda_2(r\mathbf{u}_{1:3}) = r^2\lambda_2(\mathbf{u}_{1:3})$ and $\lambda_2(\mathbf{u}_{1:3}) \leq c$, we have

$$g_{2,k}(\rho, r, \mathbf{u}_{1:3}, \mathbf{y}_{k+1:3}) \leq c \cdot r^3 e^{-\frac{1}{2}\lambda_d^{(\cup)}(r, \mathbf{u}_{1:k}, \mathbf{y}_{k+1:3})} \mathbb{1}_{r^2\lambda_2(\mathbf{u}_{1:3}) > v_\rho} \mathbb{1}_{E_{k,r,\mathbf{u}_{1:3}}}(\mathbf{y}_{k+1:3}) \tag{3.26}$$

for all $k = 0, 1, 2$. To bound $g_{2,k}(\cdot)$, the key idea is to derive a suitable lower bound for the area of the union of two disks (see Figure 1 (a)). This is provided in the following fundamental lemma.

Lemma 7. *Let* $\{\mathbf{x}_{1:3}\} = \{x_1, x_2, x_3\}$ *and* $\{\mathbf{x}'_{1:3}\} = \{x'_1, x'_2, x'_3\}$ *be two sets of three points in* $\mathbf{R}^2$ *such that* $x_i \notin B(\mathbf{x}'_{1:3})$ *and* $x'_j \notin B(\mathbf{x}_{1:3})$ *for all* $i, j = 1, 2, 3$. *Let us assume that* $R := R(\mathbf{x}_{1:3}) \geq R(\mathbf{x}'_{1:3})$. *Then*

$$\lambda_2\left(B(\mathbf{x}_{1:3}) \cup B(\mathbf{x}'_{1:3})\right) \geq \left(\frac{\pi}{2} - 1\right)R^2 + \lambda_2(\mathbf{x}_{1:3}) + \lambda_2(\mathbf{x}'_{1:3}). \tag{3.27}$$

Proof of Lemma 7. Let $\{\mathbf{x}_{1:3}\}$ and $\{\mathbf{x}'_{1:3}\}$ be fixed.

If the interior of $B(\mathbf{x}_{1:3}) \cap B(\mathbf{x}'_{1:3})$ is empty, we have

$$\lambda_2(B(\mathbf{x}_{1:3}) \cup B(\mathbf{x}'_{1:3})) = \lambda_2(B(\mathbf{x}_{1:3})) + \lambda_2(B(\mathbf{x}'_{1:3})) \geq \pi R^2 + \lambda_2(\mathbf{x}'_{1:3}). \tag{3.28}$$

Moreover, the maximal area of a triangle inscribed in a ball of radius R is $\frac{3\sqrt{3}}{4}R^2$ which is the area of an equilateral triangle. In particular, we have $\lambda_2(\mathbf{x}_{1:3}) \leq \frac{3\sqrt{3}}{4}R^2$. This together with (3.28) implies that

$$\lambda_2(B(\mathbf{x}_{1:3}) \cup B(\mathbf{x}'_{1:3})) \geq \left(\pi - \frac{3\sqrt{3}}{4}\right)R^2 + \lambda_2(\mathbf{x}_{1:3}) + \lambda_2(\mathbf{x}'_{1:3}) \geq \left(\frac{\pi}{2} - 1\right)R^2 + \lambda_2(\mathbf{x}_{1:3}) + \lambda_2(\mathbf{x}'_{1:3}).$$

If $B(\mathbf{x}_{1:3}) \cap B(\mathbf{x}'_{1:3})$ has nonempty interior, the intersection of the circumspheres induced by the points $\mathbf{x}_{1:3}$ and $\mathbf{x}'_{1:3}$ is reduced to two points, say $p_1, p_2 \in \mathbf{R}^2$. Let us denote by $\mathbf{L}$ the affine line (p_1, p_2) and $\mathbf{H}^-$ (respectively $\mathbf{H}^+$) the half plane delimited by $\mathbf{L}$ and containing (respectively not containing) the circumcenter $z(\mathbf{x}_{1:3})$. Since $x_i \notin B(\mathbf{x}'_{1:3})$ and $x'_j \notin B(\mathbf{x}_{1:3})$, $i, j = 1, 2, 3$, the triangle $\Delta(\mathbf{x}'_{1:3})$ is included in $\mathbf{H}^+$. Hence

$$\begin{aligned}\lambda_2\left(B(\mathbf{x}_{1:3}) \cup B(\mathbf{x}'_{1:3})\right) &= \lambda_2\left((B(\mathbf{x}_{1:3}) \cup B(\mathbf{x}'_{1:3})) \cap \mathbf{H}^-\right) + \lambda_2\left((B(\mathbf{x}_{1:3}) \cup B(\mathbf{x}'_{1:3})) \cap \mathbf{H}^+\right) \\ &\geq \lambda_2(B(\mathbf{x}_{1:3}) \cap \mathbf{H}^-) + \lambda_2(\mathbf{x}'_{1:3}).\end{aligned} \tag{3.29}$$

In the rest of the proof, we provide a suitable lower bound for $\lambda_2(B(\mathbf{x}_{1:3}) \cap \mathbf{H}^-)$. To do it, we denote by $\theta \in [0, 2\pi]$ the angle $\angle p_1 z(\mathbf{x}_{1:3}) p_2$. Actually $\theta \in [0, \pi]$: this comes from the fact that $\lambda_2(B(\mathbf{x}_{1:3}) \cap \mathbf{H}^-) \geq \frac{\pi}{2}R^2$ since $R := R(\mathbf{x}_{1:3}) \geq R(\mathbf{x}'_{1:3})$. The area of the cap $B(\mathbf{x}_{1:3}) \cap \mathbf{H}^-$ is given by

$$\lambda_2(B(\mathbf{x}_{1:3}) \cap \mathbf{H}^-) = \left(\pi - \frac{1}{2}(\theta - \sin\theta)\right)R^2. \tag{3.30}$$

(a) (b)

Figure 1: (a). A union of two disks. (b). The triangle which maximizes the area.

We discuss below two cases depending on θ. When $\theta \in [0, 2\pi/3]$, we deduce from (3.30) that

$$\lambda_2(B(\mathbf{x}_{1:3}) \cap \mathbf{H}^-) \geq \left(\frac{2\pi}{3} + \frac{\sqrt{3}}{4}\right) R^2. \tag{3.31}$$

Since $\lambda_2(\mathbf{x}_{1:3})$ is less than $\frac{3\sqrt{3}}{4}R^2$, we deduce from (3.31) that

$$\lambda_2(B(\mathbf{x}_{1:3}) \cap \mathbf{H}^-) \geq \lambda_2(\mathbf{x}_{1:3}) + \left(\frac{2\pi}{3} - \frac{\sqrt{3}}{2}\right) R^2 \geq \lambda_2(\mathbf{x}_{1:3}) + \left(\frac{\pi}{2} - 1\right) R^2. \tag{3.32}$$

In that case, the inequality (3.27) results from (3.29) and (3.32).

When $\theta \in [2\pi/3, \pi]$, with a standard method of geometry, we can show that the maximal area $M(\theta)$ of a triangle inscribed in $B(\mathbf{x}_{1:3}) \cap \mathbf{H}^-$ is

$$M(\theta) = \left(\sin\frac{\theta}{2} + \frac{1}{2}\sin\theta\right) R^2. \tag{3.33}$$

Actually, the triangle which maximizes the area is isoscele with central angles $\pi - \theta/2, \pi - \theta/2$ and θ (see Figure 1 (b)). We obtain from (3.30) and (3.33) that

$$\lambda_2(B(\mathbf{x}_{1:3}) \cap \mathbf{H}^-) \geq M(\theta) + \left(\frac{\pi}{2} - 1\right) R^2 + \left(\frac{\pi}{2} + 1 - \left(\frac{1}{2}\theta + \sin\frac{\theta}{2}\right)\right) R^2.$$

The last term of the right-hand side is a decreasing function on $[0, \pi]$. Its minimum equals 0 at $\theta = \pi$, i.e.

$$\frac{\pi}{2} + 1 - \left(\frac{1}{2}\theta + \sin\frac{\theta}{2}\right) \geq 0$$

for all $\theta \in [0, \pi]$. This shows that

$$\lambda_2(B(\mathbf{x}_{1:3}) \cap \mathbf{H}^-) \geq M(\theta) + \left(\frac{\pi}{2} - 1\right) R^2. \tag{3.34}$$

The inequality (3.27) results from (3.29), (3.34) and the fact that $M(\theta) \geq \lambda_2(\mathbf{x}_{1:3})$. $\square$

We can now derive an upper bound for $g_{2,k}(\cdot)$ for all $k = 0, 1, 2$. Indeed, if $\mathbf{y}_{k+1:3} \in E_{k,r,\mathbf{u}_{1:3}}$, where $E_{k,r,\mathbf{u}_{1:3}}$ has been defined in (3.7), the sets of points $\{\mathbf{x}_{1:3}\} = \{r\mathbf{u}_{1:3}\}$ and $\{\mathbf{x}'_{1:3}\} = \{r\mathbf{u}_{1:k}, \mathbf{y}_{k+1:3}\}$ satisfy the assumptions of Lemma 7 since $R(r\mathbf{u}_{1:3}) = r$ and $R(r\mathbf{u}_{1:3}) \geq R(r\mathbf{u}_{1:k}, \mathbf{y}_{k+1:3})$. Using the fact that $B(r\mathbf{u}_{1:3}) = B(0, r)$, $\lambda_2(r\mathbf{u}_{1:3}) > v_\rho$ and $\lambda_2(r\mathbf{u}_{1:k}, \mathbf{y}_{k+1:3}) > v_\rho$, we deduce from (3.8), (3.26) and (3.27) that

$$g_{2,k}(\rho, r, \mathbf{u}_{1:3}, \mathbf{y}_{k+1:3}) \leq c \cdot r^3 e^{-\frac{1}{2}\left(\left(\frac{\pi}{2}-1\right)r^2 + 2v_\rho\right)} \mathbb{1}_{r^2\lambda_2(\mathbf{u}_{1:3}) > v_\rho} \mathbb{1}_{E_{k,r,\mathbf{u}_{1:3}}}(\mathbf{y}_{k+1:3}).$$

Since $\frac{3\sqrt{3}}{4}r^2 \geq r^2\lambda_2(\mathbf{u}_{1:3})$ and $\alpha_2 = \frac{2\pi}{3\sqrt{3}}$, we deduce from (3.25) that $r^2\lambda_2(\mathbf{u}_{1:3}) > v_\rho$ implies $r > (2(\log\rho + c')/\pi)^{1/2}$, where $c' = \log(3/2) + t$. Integrating the right-hand side with respect to $\mathbf{y}_{k+1:3}$, we obtain

$$G_{2,k}(\rho) \leq c \cdot \rho \int_{(2(\log\rho+c')/\pi)^{1/2}}^{\infty} \int_{(\mathbf{S}^1)^3} r^3 e^{-\frac{1}{2}\left(\left(\frac{\pi}{2}-1\right)r^2+2v_\rho\right)} \times \lambda_{2(3-k)}(E_{k,r,\mathbf{u}_{1:3}}) d\sigma(\mathbf{u}_{1:3}) dr. \tag{3.35}$$

The following lemma gives a uniform upper bound for $\lambda_{2(3-k)}(E_{k,r,\mathbf{u}_{1:3}})$.

Lemma 8. *Let $\mathbf{u}_{1:3} \in (\mathbf{S}^1)^3$ and $r > (2(\log\rho + c')/\pi)^{1/2}$. Then for ρ large enough*

$$\lambda_{2(3-k)}(E_{k,r,\mathbf{u}_{1:3}}) \leq c \cdot r^{2(3-k)}. \tag{3.36}$$

Proof of Lemma 8. We discuss the three cases $k = 0, 1, 2$.

If $k = 2$, we show that $E_{2,r,\mathbf{u}_{1:3}}$ is included in a ball of radius r up to a multiplicative constant and centered at 0. Let y_3 be in $E_{2,r,\mathbf{u}_{1:3}}$. From the triangle inequality, we have

$$|y_3| \leq |y_3 - z(r\mathbf{u}_{1:2}, y_3)| + |z(r\mathbf{u}_{1:2}, y_3)| \leq r + \operatorname{diam}(\mathfrak{C}_\rho). \tag{3.37}$$

The last inequality comes from the fact that $|y_3 - z(r\mathbf{u}_{1:2}, y_3)|$ is the circumradius of $\Delta(r\mathbf{u}_{1:2}, y_3)$, which is less than r, and the fact that $z(r\mathbf{u}_{1:2}) \in \mathfrak{C}_\rho$. Moreover

$$\operatorname{diam}(\mathfrak{C}_\rho) \leq c \cdot (\log\rho)^{1/2} \leq c \cdot r, \tag{3.38}$$

where the last inequality holds for ρ large enough since $r > (2(\log\rho + c')/\pi)^{1/2}$ converges to infinity as ρ goes to infinity. We deduce from (3.37) and (3.38) that $|y_3| \leq c \cdot r$. This shows that $E_{2,r,\mathbf{u}_{1:3}} \subset B(0, c \cdot r)$. In particular, we get $\lambda_2(E_{2,r,\mathbf{u}_{1:3}}) \leq c \cdot r^2$.

If $k = 1$ or $k = 0$, proceeding along the same lines as in the case $k = 2$, we show that $E_{k,r,\mathbf{u}_{1:3}} \subset B(0, c \cdot r)^{3-k}$ and consequently we get $\lambda_{2(3-k)}(E_{k,r,\mathbf{u}_{1:3}}) \leq c \cdot r^{2(3-k)}$.

□

We can now derive an upper bound for $G_{2,k}(\rho)$. Indeed, integrating $\mathbf{u}_{1:3}$ over $(\mathbf{S}^1)^3$, we deduce from (3.35) and (3.36) that

$$G_{2,k}(\rho) \leq c \cdot \rho \int_{(2(\log\rho+c')/\pi)^{1/2}}^{\infty} r^{9-k} e^{-\frac{1}{2}\left(\left(\frac{\pi}{2}-1\right)r^2+2v_\rho\right)} dr.$$

Integrating the right-hand side, we obtain from (3.25) that

$$G_{2,k}(\rho) \leq c \cdot (\log\rho)^{8-2k} \rho^{(\pi+2-3\sqrt{3})/2\pi} = O\left((\log\rho)^8 \rho^{-\epsilon}\right) \tag{3.39}$$

with $\epsilon = -\pi - 2 + 3\sqrt{3} > 0$. Proposition 2 results from (3.39), Lemma 6 and Theorem 1. □

Lemma 7 provides the main tool of the proof. We can note that inequality (3.27) is obvious when we replace $\frac{\pi}{2} - 1$ by a constant $\alpha \leq \pi - \frac{3\sqrt{3}}{2}$. Indeed, if $\Delta(\mathbf{x}_{1:3})$ and $\Delta(\mathbf{x}'_{1:3})$ are two triangles with $R := R(\mathbf{x}_{1:3}) \geq R(\mathbf{x}'_{1:3})$, a trivial inequality is $\lambda_2(B(\mathbf{x}_{1:3}) \cup B(\mathbf{x}'_{1:3})) \geq \pi R^2$. Consequently

$$\lambda_2(B(\mathbf{x}_{1:3}) \cup B(\mathbf{x}'_{1:3})) \geq \left(\pi - \frac{3\sqrt{3}}{2}\right) R^2 + \lambda_2(\mathbf{x}_{1:3}) + \lambda_2(\mathbf{x}'_{1:3})$$

since $\lambda_2(\mathbf{x}_{1:3})$ and $\lambda_2(\mathbf{x}'_{1:3})$ are less than $\frac{3\sqrt{3}}{4}R^2$. Nevertheless, the previous lower bound is not enough to guarantee that $G_{2,k}(\rho)$ converges to 0. The important fact in Lemma 7 is that we consider the more precise constant $\frac{\pi}{2} - 1 > \pi - \frac{3\sqrt{3}}{2}$.

Another remark deals with the shape of the cell maximizing the area. Indeed, the maximum of circumradii of a planar Poisson-Delaunay tessellation, denoted by $R_{\max,PDT}(\rho)$, satisfies (see (2c) in [8]):

$$\mathbb{P}\left(\frac{\pi}{2}R_{\max,PDT}(\rho)^2 - \log\left(\rho\log\rho\right) \leq t\right) \underset{\rho\to\infty}{\longrightarrow} e^{-e^{-t}}$$

for all $t \in \mathbf{R}$. This shows that the expectation of $\frac{3\sqrt{3}}{4}R_{\max,PDT}(\rho)^2$, which is the area of an equilateral triangle of circumradius $R_{\max,PDT}(\rho)$, is of order $\frac{3\sqrt{3}}{2\pi}\log\rho$. According to (3.21), this is also the order of $\mathbb{E}\left[A_{\max,PDT}(\rho)\right]$. It seems that the shape of the cell maximizing the area tends to that of an equilateral triangle. This fact can be connected to the D.G. Kendall's conjecture and to the work of Hug and Schneider in [13].

3.3 Minimum of the areas, $d = 2$

In our third example, we calculate the asymptotic behaviour of the minimum of the areas of the cells of a Poisson-Delaunay tessellation (of intensity 1) in $\mathbf{R}^2$, i.e.

$$A_{\min,PDT}(\rho) = \min_{\substack{C\in\mathfrak{m}_{PDT},\\ z(C)\in\mathbf{W}_\rho}} \lambda_2(C).$$

Proposition 6. *Let $\mathfrak{m}_{PDT}$ be a Poisson-Delaunay tessellation of intensity $\gamma = 1$ in $\mathbf{R}^2$. Then for all $t \geq 0$*

$$\mathbb{P}\left(\alpha_3^{3/5}\rho^{3/5}A_{\min,PDT}(\rho) \geq t\right) \underset{\rho\to\infty}{\longrightarrow} e^{-t^{5/3}}, \tag{3.40}$$

where $\alpha_3 = 2^{-2/3}\cdot 3^{-1/2}\cdot 5^{-1}\cdot \pi^{2/3}\cdot\Gamma(1/6)^2$.

In [36], Schulte and Thäle investigate the behaviour of the smallest area S_ρ of all triangles that can be formed by three points of the Poisson point process, i.e.

$$S_\rho = \min_{\substack{\mathbf{x}_{1:3}\in\mathbf{X}^3,\\ z(\mathbf{x}_{1:3})\in\mathbf{W}_\rho}} \lambda_2(\mathbf{x}_{1:3}).$$

The asymptotic behaviour of S_ρ (see Theorem 2.5. in [36]) is given by $\mathbb{P}\left(\rho S_\rho \geq t\right) \underset{\rho\to\infty}{\longrightarrow} e^{-\beta t}$, where β is a constant which can be made explicit. The previous limit compared to (3.40) shows that the smallest area of the Delaunay cells is much larger than the smallest area of all triangles.

Proof of Proposition 6. First, we calculate the asymptotic behaviour of the distribution function of $\lambda_2(\mathcal{C})$ (see (3.22)). A Taylor expansion of $K_{1/6}(x)$ as x goes to 0 is given by (see Formula 9.6.9, p. 375 in [1])

$$K_{1/6}(x) = 2^{-5/6}\Gamma\left(1/6\right)x^{-1/6} + o(x^{-1/6}).$$

This together with (3.20) and (3.22) shows that

$$\mathbb{P}\left(\lambda_2(\mathcal{C}) < v\right) = \frac{6}{\pi}\cdot 2^{-5/3}\Gamma\left(1/6\right)^2\int_0^{\alpha_2\beta_2 v}\left(x^{2/3} + o\left(x^{2/3}\right)\right)dx = \alpha_3\cdot v^{5/3} + o\left(v^{5/3}\right). \tag{3.41}$$

Taking for all $t \geq 0$

$$v_\rho = v_\rho(t) = (\alpha_3^{-1}\rho^{-1})^{3/5}t \tag{3.42}$$

we obtain

$$G_1(\rho) = |\rho\mathbb{P}\left(\lambda_2(\mathcal{C}) < v_\rho\right) - t^{5/3}| \underset{\rho\to\infty}{\longrightarrow} 0. \tag{3.43}$$

We investigate below the rate of convergence of $G_2(\rho)$. Taking $f(r\mathbf{u}_{1:3}) = r^2\lambda_2(\mathbf{u}_{1:3})$ and using the fact that $\lambda_2(B(0,r)\cup B(r\mathbf{u}_{1:k},\mathbf{y}_{k+1:3}))$ is greater than πr^2, for all $k=0,1,2$, we have

$$g_{2,k}(\rho,r,\mathbf{u}_{1:3},\mathbf{y}_{k+1:3}) \leq r^3 e^{-\frac{1}{2}\pi r^2}\lambda_2(\mathbf{u}_{1:3})\mathbb{1}_{r^2\lambda_2(\mathbf{u}_{1:3})<v_\rho}\mathbb{1}_{E_{k,r,\mathbf{u}_{1:3}}}$$

according to (3.8) and (3.11). Integrating with respect to $\mathbf{y}_{1:3}$, this gives

$$G_{2,k}(\rho) \leq \rho\int_0^\infty\int_{(\mathbf{S}^1)^3} r^3 e^{-\frac{1}{2}\pi r^2}\lambda_2(\mathbf{u}_{1:3})\lambda_{2(3-k)}(E_{k,r,\mathbf{u}_{1:3}})\mathbb{1}_{r^2\lambda_2(\mathbf{u}_{1:3})<v_\rho}d\sigma(\mathbf{u}_{1:3})dr. \tag{3.44}$$

As in the proof of Proposition 5, we derive a suitable upper bound for the volume of $E_{k,r,\mathbf{u}_{1:3}}$.

Lemma 9. *Let $\mathbf{u}_{1:3}\in(\mathbf{S}^1)^3$ such that $u_1\neq u_2$ and $r>0$. Then*

$$\lambda_2(E_{2,r,\mathbf{u}_{1:3}}) \leq c\cdot v_\rho|u_1-u_2|^{-1}, \tag{3.45a}$$

$$\lambda_4(E_{1,r,\mathbf{u}_{1:3}}) \leq c\cdot r^2 v_\rho, \tag{3.45b}$$

$$\lambda_6(E_{0,r,\mathbf{u}_{1:3}}) \leq c\cdot\log\rho\cdot r^2 v_\rho. \tag{3.45c}$$

Proof of Lemma 9. Let y_3 be in $E_{2,r,\mathbf{u}_{1:3}}$. Since $R(r\mathbf{u}_{1:2},y_3)$ is less than r, we have $|y_3-ru_1|\leq 2R(r\mathbf{u}_{1:2},y_3)\leq 2r$. In particular, we obtain

$$|y_3|\leq 3r. \tag{3.46}$$

Moreover, the area of the triangle $\Delta(r\mathbf{u}_{1:2},y_3)$ is given by

$$\lambda_2(r\mathbf{u}_{1:2},y_3) = \frac{1}{2}r|u_1-u_2|\cdot\delta(y_3,\mathbf{L}(ru_1,ru_2)), \tag{3.47}$$

where $\mathbf{L}(ru_1,ru_2)$ is the line through $p_1=ru_1$, $p_2=ru_2$ and where $\delta(y_3,\mathbf{L}(ru_1,ru_2))$ denotes the distance between this line and the point y_3. Since $\lambda_2(r\mathbf{u}_{1:2},y_3)<v_\rho$, it results from (3.47) that

$$\delta(y_3,\mathbf{L}(ru_1,ru_2)) \leq \frac{2v_\rho}{r|u_1-u_2|}. \tag{3.48}$$

The inequalities (3.46) and (3.48) show that $E_{2,r,\mathbf{u}_{1:3}}$ is included in the intersection of a ball of radius $3r$ and a strip of width $\frac{4v_\rho}{r|u_1-u_2|}$, i.e.

$$\lambda_2(E_{2,r,\mathbf{u}_{1:3}}) \leq 6r\times\frac{4v_\rho}{r|u_1-u_2|} = c\cdot v_\rho|u_1-u_2|^{-1}.$$

Secondly, we bound $\lambda_4(E_{1,r,\mathbf{u}_{1:3}})$. Taking the (spherical coordinates type) change of variables $\phi_2:\mathbf{R}_+\times\mathbf{S}^1\to\mathbf{R}^2, (s',u_2')\mapsto y_2=ru_1+s'u_2'$ with Jacobian $|D\phi_2(s',u_2')|=s'$, we obtain

$$\lambda_4(E_{1,r,\mathbf{u}_{1:3}}) \leq \int_0^{2r}\int_{\mathbf{S}^1}\int_{\mathbf{R}^2} s'\mathbb{1}_{\lambda_2(ru_1,ru_1+s'u_2',y_3)<v_\rho}\mathbb{1}_{R(ru_1,ru_1+s'u_2',y_3)\leq r}dy_3 d\sigma(u_2')ds'. \tag{3.49}$$

The positive number s' is integrated on $(0,2r]$. Indeed, the inequality $R(ru_1,ru_1+s'u_2',y_3)\leq r$ implies that $s'=|(ru_1+s'u_2')-ru_1|\leq 2r$. Proceeding along the same lines as in the proof of (3.45a), we show that y_3 belongs to the ball $B(0,3r)$ and a strip of width $\frac{4v_\rho}{s'}$. Integrating (3.49) with respect to y_3, we deduce that

$$\lambda_4(E_{1,r,\mathbf{u}_{1:3}}) \leq 24\int_0^{2r}\int_{\mathbf{S}^1} v_\rho r d\sigma(u_2')ds' = c\cdot r^2 v_\rho.$$

Finally, we bound $\lambda_6(E_{0,r,\mathbf{u}_{1:3}})$. Taking the same change of variables as in (3.12), we have

$$\lambda_6(E_{0,r,\mathbf{u}_{1:3}}) \leq \int_{(\mathbf{R}^2)^3} \mathbb{1}_{z(\mathbf{y}_{1:3})\in\mathfrak{C}_\rho}\mathbb{1}_{R(\mathbf{y}_{1:3})<r}\mathbb{1}_{\lambda_2(\mathbf{y}_{1:3})<v_\rho} d\mathbf{y}_{1:3}$$
$$= \int_{\mathfrak{C}_\rho}\int_0^r\int_{(\mathbf{S}^1)^3} r'^3\lambda_2(\mathbf{u}'_{1:3})\mathbb{1}_{r'^2\lambda_2(u'_1,u'_2,u'_3)<v_\rho} d\sigma(\mathbf{u}'_{1:3})dr'dz'.$$

Bounding $r'^3\lambda_2(u'_1,u'_2,u'_3)$ by $r'v_\rho$ and integrating with respect to $z'\in\mathfrak{C}_\rho$, $r'\in[0,r]$ and $\mathbf{u}'_{1:3}\in(\mathbf{S}^1)^3$, we show that $\lambda_6(E_{0,r,\mathbf{u}_{1:3}})$ is less than $c\cdot\lambda_2(\mathfrak{C}_\rho)r^2v_\rho$ with $\lambda_2(\mathfrak{C}_\rho)\leq c\cdot\log\rho$. □

We can now derive a suitable upper bound for $G_{2,k}(\rho)$. Indeed, if $k=0$, we deduce from (3.44) and (3.45c) that

$$G_{2,0}(\rho) \leq c\cdot\log\rho\cdot\rho v_\rho\int_0^\infty\int_{(\mathbf{S}^1)^3} r^5e^{-\frac{1}{2}\pi r^2}\lambda_2(\mathbf{u}_{1:3})\mathbb{1}_{r^2\lambda_2(\mathbf{u}_{1:3})<v_\rho}d\sigma(\mathbf{u}_{1:3})dr$$
$$\leq c\cdot\log\rho\cdot\rho v_\rho^2\int_0^\infty\int_{(\mathbf{S}^1)^3} r^3e^{-\frac{1}{2}\pi r^2}d\sigma(\mathbf{u}_{1:3})dr.$$

First, we notice that the integral of the right-hand side is bounded. Besides, (3.42) shows that $G_{2,0}(\rho)$ is less than $c\cdot\log\rho\cdot\rho^{-1/5}$. In the same spirit, when $k=1$, we obtain that $G_{2,1}(\rho)\leq c\cdot\rho^{-1/5}$ according to (3.44) and (3.45b). Hence

$$G_{2,0}(\rho) = O\left(\log\rho\cdot\rho^{-1/5}\right) \text{ and } G_{2,1}(\rho) = O\left(\rho^{-1/5}\right). \tag{3.50}$$

Finally, if $k=2$, we deduce from (3.44) and (3.45a) that

$$G_{2,2}(\rho) \leq c\cdot\rho v_\rho\int_0^\infty\int_{(\mathbf{S}^1)^3} r^3e^{-\frac{1}{2}\pi r^2}\lambda_2(\mathbf{u}_{1:3})|u_1-u_2|^{-1}\mathbb{1}_{r^2\lambda_2(\mathbf{u}_{1:3})<v_\rho}\mathbb{1}_{u_1\neq u_2}d\sigma(\mathbf{u}_{1:3})dr.$$

Let ϕ_3 be the change of variables

$$\phi_3 : [0,2\pi)^3 \longrightarrow (\mathbf{S}^1)^3$$
$$\boldsymbol{\theta}_{1:3} \longmapsto \mathbf{u}_{1:3} \text{ with } u_1 = u(-\theta_1+\theta_3), u_2 = u(\theta_1+\theta_3) \text{ and } u_3 = u(\theta_2+\theta_3),$$

where $u(\theta)=(\cos\theta,\sin\theta)$. For all $\boldsymbol{\theta}_{1:3}\in(0,2\pi)\times[0,2\pi)^2$, let us denote by $A(\boldsymbol{\theta}_{1:3})=\lambda_2(\mathbf{u}_{1:3})$ with $\mathbf{u}_{1:3}=\phi_3(\boldsymbol{\theta}_{1:3})$. Since $|u_1-u_2|=2|\sin\theta_1|$, we have

$$G_{2,2}(\rho) \leq c\cdot\rho v_\rho\int_0^\infty\int_{(0,\pi/2)\times[0,2\pi)^2} r^3e^{-\frac{1}{2}\pi r^2}A(\boldsymbol{\theta}_{1:3})|\sin\theta_1|^{-1}\mathbb{1}_{r^2A(\boldsymbol{\theta}_{1:3})<v_\rho}d\boldsymbol{\theta}_{1:3}dr.$$

Without loss of generality, we have assumed that θ_1 belongs to $(0,\pi/2)$. Let $\epsilon>\frac{3}{5}$ be fixed. The previous inequality can be written as

$$G_{2,2}(\rho) \leq c\cdot\rho v_\rho\int_0^\infty\int_{(0,\rho^{-\epsilon}[\times[0,2\pi)^2} r^3e^{-\frac{1}{2}\pi r^2}A(\boldsymbol{\theta}_{1:3})|\sin\theta_1|^{-1}\mathbb{1}_{r^2A(\boldsymbol{\theta}_{1:3})<v_\rho}d\boldsymbol{\theta}_{1:3}dr$$
$$+ c\cdot\rho v_\rho\int_0^\infty\int_{[\rho^{-\epsilon},\pi/2)\times[0,2\pi)^2} r^3e^{-\frac{1}{2}\pi r^2}A(\boldsymbol{\theta}_{1:3})|\sin\theta_1|^{-1}\mathbb{1}_{r^2A(\boldsymbol{\theta}_{1:3})<v_\rho}d\boldsymbol{\theta}_{1:3}dr = G_{2,2}^{(1)}(\rho)+G_{2,2}^{(2)}(\rho), \tag{3.51}$$

where $G_{2,2}^{(1)}(\rho)$ and $G_{2,2}^{(2)}(\rho)$ denote respectively the first and the second terms of the right-hand side. Let us note that $A(\boldsymbol{\theta}_{1:3})|\sin\theta_1|^{-1}$ is bounded since, according to (3.47), we have $A(\boldsymbol{\theta}_{1:3})=\frac{1}{2}\cdot 2|\sin\theta_1|\cdot\delta(u_3,\mathbf{L}(\mathbf{u}_{1:2}))$, where $\mathbf{u}_{1:3}=\phi_3(\boldsymbol{\theta}_{1:3})$ and $\delta(u_3,\mathbf{L}(\mathbf{u}_{1:2}))\leq 2$. Hence, the first integral of the right-hand side of (3.51) is less than

$$G_{2,2}^{(1)}(\rho) \leq c\cdot\rho v_\rho\int_0^\infty\int_{(0,\rho^{-\epsilon})\times[0,2\pi)^2} r^3e^{-\frac{1}{2}\pi r^2}d\boldsymbol{\theta}_{1:3}dr \leq c\cdot\rho^{1-\epsilon}v_\rho = O(\rho^{-1/5}) \tag{3.52}$$

since $v_\rho = c \cdot \rho^{-3/5}$ and $\epsilon > \frac{3}{5}$. Moreover, bounding $A(\boldsymbol{\theta}_{1:3})$ by $r^{-2}v_\rho$ in the second integral of (3.51), we have

$$G_{2,2}^{(2)}(\rho) \leq c \cdot \rho v_\rho^2 \int_0^\infty \int_{[\rho^{-\epsilon},\pi/2)\times[0,2\pi)^2} re^{-\frac{1}{2}\pi r^2} |\sin\theta_1|^{-1} d\boldsymbol{\theta}_{1:3} dr \leq c \cdot \log\rho \cdot \rho v_\rho^2 = O\left(\log\rho \cdot \rho^{-1/5}\right) \tag{3.53}$$

since $\int_{\rho^{-\epsilon}}^{\pi/2} \frac{1}{|\sin\theta_1|} d\theta_1$ is of order $\log\rho$.

From (3.50), (3.51), (3.52) and (3.53), we deduce that $G_2(\rho) = O\left(\log\rho \cdot \rho^{-1/5}\right)$. Proposition 6 is now a direct consequence of (3.43) and Theorem 1. □

Remark 4. 1. To the best of our knowledge, there is no more accurate result on the Taylor expansion of $K_{1/6}(\cdot)$ which could provide the rate of convergence of $\mathbb{P}\left(\lambda_2(\mathcal{C}) < v\right)$. Actually, the rate of convergence can be investigated by using the following expression of the density of $\lambda_2(\mathcal{C})$ due to Rathie [31]:

$$f(x) = \sum_{k=0}^{\infty} c_{k,1} x^{2/3+2k} + \sum_{k=0}^{\infty} c_{k,2} x^{1+2k} + \sum_{k=0}^{\infty} c_{k,3} x^{4/3+2k}$$

for some constants $c_{k,i}$, $k \geq 0, 1 \leq i \leq 3$. It results from a Taylor expansion that $\mathbb{P}\left(\lambda_2(\mathcal{C}) < v\right) = c_{0,1} \cdot v^{5/3} + O(v^2)$ for all $v \geq 0$. Taking $v = v_\rho$ as in (3.42), we obtain $G_1(\rho) = |\rho\mathbb{P}\left(\lambda_2(\mathcal{C}) < v_\rho\right) - t^{5/3}| = O\left(\rho^{-1/5}\right)$ so that

$$\left|\mathbb{P}\left(\alpha_3^{3/5}\rho^{3/5} A_{PDT,\min}(\rho) \geq t\right) - e^{-t^{5/3}}\right| = O\left(\log\rho \cdot \rho^{-1/5}\right).$$

2. When $d \geq 3$, we can show that the density of $\lambda_d(\mathcal{C})$ (see (2.5) in [31])) satisfies $f(x) = c \cdot x + o(x)$. This implies $\rho\mathbb{P}\left(\lambda_d(\mathcal{C}) < c \cdot \rho^{-1/2} t\right) \underset{\rho\to\infty}{\longrightarrow} t^2$. Unfortunately, the same method as in the proof of Proposition 6 is not enough to show that $G_2(\rho)$ converges to 0.

4 Extreme Values of a Poisson-Voronoi tessellation

Let χ be a locally finite subset of $\mathbf{R}^d$. For all $x \in \chi$, we denote by $C_\chi(x)$ the Voronoi cell of nucleus x defined as

$$C_\chi(x) = \{y \in \mathbf{R}^d : |x - y| \leq |x' - y|, x' \in \chi\}.$$

For all $x \in \chi$, we denote by $\mathcal{N}_\chi(x)$ the set of neighbors of x and by $N_\chi(x)$ its cardinality, i.e.

$$\mathcal{N}_\chi(x) = \{x' \in \chi, C_\chi(x') \cap C_\chi(x) \neq \varnothing\} \text{ and } N_\chi(x) = \#\mathcal{N}_\chi(x).$$

We also consider the two quantities

$$D(C_\chi(x)) = \max_{x' \in \mathcal{N}_\chi(x)} |x - x'|, \text{ and } \mathcal{F}(C_\chi(x)) = \bigcup_{y \in C_\chi(x)} B(y, |y - x|)$$

which are the distance to the farthest neighbor and the so-called Voronoi flower of nucleus $x \in \chi$. A Voronoi tessellation corresponds to the dual graph of Delaunay tessellation in the following way: there exists an edge between two points $x, x' \in \chi$ in the Delaunay graph if and only if they are Voronoi neighbors, i.e. $C_\chi(x) \cap C_\chi(x') \neq \varnothing$.

When $\chi = \mathbf{X}$ is a Poisson point process (of intensity 1), the family $\mathfrak{m}_{PVT} = \{C_\mathbf{X}(x), x \in \mathbf{X}\}$ is called the Poisson-Voronoi tessellation. Such a model is extensively used in many domains such as cellular biology [29], astrophysics [30], telecommunications [4] and ecology [33]. For a complete account, we refer to the books [23, 25, 35] and the survey [7].

As in Section 4, the window $\mathbf{W}_\rho = \rho^{1/d}[0,1]^d$ is partitioned into $N_\rho = \left\lfloor \left(\frac{\rho}{2\log \rho}\right)^{1/d} \right\rfloor^d$ sub-cubes $\mathbf{i} \in V_\rho$. The event A_ρ is the same as in (3.2) and we can show that it satisfies CONDITION (FRC) for the Poisson-Voronoi tessellation with arguments very similar to the proof of Lemma 5. For each cell $C = C_{\mathbf{X}}(x) \in \mathfrak{m}_{PVT}$, we take $z(C_{\mathbf{X}}(x)) = x$. A consequence of Slivnyak-Mecke theorem (see e.g. Theorem 3.3.5 in [35]) shows that the typical cell $\mathcal{C}$ satisfies the equality in distribution $\mathcal{C} \overset{\mathcal{D}}{=} C_{\mathbf{X}\cup\{0\}}(0)$. Besides, the function $G_2(\cdot)$ defined in (1.6) has an integral representation. Indeed, from Slivnyak-Mecke formula, it can be written as

$$G_2(\rho) = \rho \int_{\mathfrak{C}_\rho} \mathbb{P}\left(f(C_{\mathbf{X}\cup\{0,y\}}(0)) > v_\rho, f(C_{\mathbf{X}\cup\{0,y\}}(y)) > v_\rho\right) dy. \tag{4.1}$$

Extremes of characteristic radii of Poisson-Voronoi tessellation are studied in [8]. In this paper, we give the asymptotic behaviours of two new geometrical characteristics:

$$D_{\min,PVT}(\rho) = \min_{x\in\mathbf{X}\cap\mathbf{W}_\rho} D(C_{\mathbf{X}}(x)) \text{ and } F_{\min,PVT}(\rho) = \min_{x\in\mathbf{X}\cap\mathbf{W}_\rho} \lambda_d(\mathcal{F}(C_{\mathbf{X}}(x))).$$

Obviously, $2^{-d}\kappa_d D^d_{\min,PVT}(\rho) \le \kappa_d \min_{x\in\mathbf{X}\cap\mathbf{W}_\rho} R(C_{\mathbf{X}}(x))^d \le F_{\min,PVT}(\rho)$. Actually, the following proposition shows that the two random variables $D^d_{\min,PVT}(\rho)$ and $F_{\min,PVT}(\rho)$ are of same order when ρ goes to infinity.

Proposition 7. *Let $\mathfrak{m}_{PVT}$ be a Poisson-Voronoi tessellation of intensity $\gamma = 1$. For all $t \ge 0$, we have*

$$\left|\mathbb{P}\left(\alpha_{d,4}^{1/(d+1)}\rho^{1/(d+1)} D^d_{\min,PVT}(\rho) \ge t\right) - e^{-t^{d+1}}\right| = O\left(\rho^{-1/(d+1)}\right), \tag{4.2a}$$

$$\left|\mathbb{P}\left(\alpha_{d,5}^{1/(d+1)}\rho^{1/(d+1)} F_{\min,PVT}(\rho) \ge t\right) - e^{-t^{d+1}}\right| = O\left(\rho^{-1/(d+1)}\right), \tag{4.2b}$$

where $\alpha_{d,4}$ and $\alpha_{d,5}$ are defined in (4.7) *and* (4.17) *respectively.*

Before proving Proposition 7, we need a practical lemma which is a new version of Lemma 3 in [8] adapted to our framework.

Lemma 10. *Let $v \ge 0$, $y \ne 0 \in \mathbf{R}^d$ and $\chi \subset \mathbf{R}^d$ locally finite such that $\chi \cup \{0,y\}$ is in general position, i.e. each subset of size $n<d+1$ is affinely independent (see [41]). Let us assume that each Voronoi cell associated with the set $\chi \cup \{0,y\}$ is bounded and that*

$$\mathcal{N}_{\chi\cup\{0,y\}}(0) \subset B(0,v) \text{ and } \mathcal{N}_{\chi\cup\{0,y\}}(y) \subset B(y,v). \tag{4.3}$$

Then

$$\#\left(\chi \cap (B(0,v) \cup B(y,v))\right) \ge d+1.$$

Proof of Lemma 10. Let us define $\chi_{0,y}$ as the (finite) subset

$$\chi_{0,y} = \chi \cap (B(0,v) \cup B(y,v)).$$

Thanks to (4.3), we have $C_{\chi\cup\{0,y\}}(0) = C_{\chi_{0,y}\cup\{0,y\}}(0)$ and $C_{\chi\cup\{0,y\}}(y) = C_{\chi_{0,y}\cup\{0,y\}}(y)$. In particular, this shows that the cells $C_{\chi_{0,y}\cup\{0,y\}}(0)$ and $C_{\chi_{0,y}\cup\{0,y\}}(y)$ are bounded. Hence 0 and y are in the convex hulls of $\chi_{0,y} \cup \{y\}$ and $\chi_{0,y} \cup \{0\}$ respectively (see Property **V**2, p. 58 in [25]). This implies $\{0,y\} \subset \mathrm{conv}(\chi_{0,y})$. Since $\chi \cup \{0,y\}$ is in general position, this shows that $\mathrm{conv}(\chi_{0,y})$ has a nonempty interior and consequently this proves Lemma 10. $\square$

We can now prove Proposition 7.

Proof of Proposition 7.

Proof of (4.2a)**.** To find a function $v_\rho = v_\rho(t)$ such that $G_1(\rho) = |\rho \mathbb{P}\left(D(\mathcal{C}) > v_\rho\right) - t|$ converges to 0, we have to approximate the tail of the distribution function of $D(\mathcal{C})$. First, we notice that for all $v > 0$ we have

$$D\left(C_\chi(x)\right) < v \Longleftrightarrow \mathcal{N}_\chi(x) \subset B(0,v), x \in \chi \tag{4.4}$$

for any locally finite subset χ of $\mathbf{R}^d$. This fact applied to $\chi = \mathbf{X} \cup \{0\}$ shows that

$$\mathbb{P}\left(D(\mathcal{C}) < v\right) = \sum_{k=d+1}^{\infty} \mathbb{P}\left(\mathcal{N}_{\mathbf{X}\cup\{0\}}(0) \subset B(0,v), N_{\mathbf{X}\cup\{0\}}(0) = k\right) \tag{4.5}$$

since $\mathcal{C} = C_{\mathbf{X}\cup\{0\}}(0)$. An integral representation of the right-hand side is given by (see Proposition 1 in [6])

$$\mathbb{P}\left(\mathcal{N}_{\mathbf{X}\cup\{0\}}(0) \subset B(0,v), N_{\mathbf{X}\cup\{0\}}(0) = k\right) = \frac{1}{k!}\int_{B(0,v)^k} e^{-\lambda_d(\mathcal{F}(C_{\{\mathbf{x}_{1:k}\}\cup\{0\}}(0)))}\mathbb{1}_{F_k}(\mathbf{x}_{1:k})d\mathbf{x}_{1:k},$$

where

$$F_k = \left\{\mathbf{x}_{1:k} = (x_1, \dots, x_k) \in (\mathbf{R}^d)^k, C_{\{\mathbf{x}_{1:k}\}\cup\{0\}}(0) \text{ is a convex polytope with } k \text{ facets}\right\}.$$

We recall that $\{\mathbf{x}_{1:k}\} \cup \{0\} = \{x_1, x_2, \dots, x_k, 0\}$. Taking the change of variables $x_i = vx_i'$, we obtain for all $k \geq d+1$

$$\mathbb{P}\left(\mathcal{N}_{\mathbf{X}\cup\{0\}}(0) \subset B(0,v), N_{\mathbf{X}\cup\{0\}}(0) = k\right) = v^{dk} \cdot \frac{1}{k!}\int_{B(0,1)^k} e^{-v^d\lambda_d(\mathcal{F}(C_{\{\mathbf{x}'_{1:k}\}\cup\{0\}}(0)))}\mathbb{1}_{F_k}(\mathbf{x}'_{1:k})d\mathbf{x}'_{1:k}. \tag{4.6}$$

If $k = d+1$, the previous probability is equivalent to $\alpha_{d,4} \cdot v^{d(d+1)}$ when v goes to 0, where

$$\alpha_{d,4} = \frac{1}{(d+1)!}\int_{B(0,1)^{d+1}} \mathbb{1}_{F_{d+1}}(\mathbf{x}'_{1:d+1})d\mathbf{x}'_{1:d+1}. \tag{4.7}$$

If $k \geq d+2$, the right-hand side of (4.6) is less than $\frac{\kappa_d^k}{k!}v^{dk}$ thanks to $\mathbb{1}_{F_k} \leq 1$ and $e^{-\lambda_d(\mathcal{F}(C_{\{\mathbf{x}'_{1:k}\}\cup\{0\}}(0)))} \leq 1$. It follows from (4.5) that

$$\left|\mathbb{P}\left(D(\mathcal{C}) < v\right) - \alpha_{d,4} \cdot v^{d(d+1)}\right| \leq \sum_{k=d+2}^{\infty} \frac{\kappa_d^k}{k!}v^{dk} = O(v^{d(d+2)}). \tag{4.8}$$

Now, we can choose a suitable function v_ρ. Indeed, let $t \geq 0$ be fixed and let us denote by

$$v_\rho = v_\rho(t) = \left(\alpha_{d,4}^{-1}\rho^{-1}\right)^{1/d(d+1)} t^{1/d}. \tag{4.9}$$

According to (4.8), we have

$$G_1(\rho) = |\rho\mathbb{P}\left(D(\mathcal{C}) < v_\rho\right) - t^{d+1}| = O\left(\rho^{-1/(d+1)}\right). \tag{4.10}$$

Let us give now an upper bound for the function $G_2(\rho)$ defined in (1.6). According to (4.1) and (4.4), we obtain

$$\begin{aligned} G_2(\rho) &= \rho\int_{\mathfrak{C}_\rho} \mathbb{P}\left(D(C_{\mathbf{X}\cup\{0,y\}}(0)) < v_\rho, D(C_{\mathbf{X}\cup\{0,y\}}(y)) < v_\rho\right)dy \\ &= \rho\int_{\mathfrak{C}_\rho} \mathbb{P}\left(\mathcal{N}_{\mathbf{X}\cup\{0,y\}}(0) \subset B(0,v_\rho), \mathcal{N}_{\mathbf{X}\cup\{0,y\}}(y) \subset B(y,v_\rho)\right)dy. \end{aligned} \tag{4.11}$$

To guarantee the independence of the events considered in (4.11) for each cells which are distant enough, we write

$$G_2(\rho) = \rho \int_{\mathfrak{C}_\rho \cap B(0,2v_\rho)^c} \mathbb{P}\left(\mathcal{N}_{\mathbf{X}\cup\{0,y\}}(0) \subset B(0,v_\rho), \mathcal{N}_{\mathbf{X}\cup\{0,y\}}(y) \subset B(y,v_\rho)\right) dy$$
$$+ \rho \int_{\mathfrak{C}_\rho \cap B(0,2v_\rho)} \mathbb{P}\left(\mathcal{N}_{\mathbf{X}\cup\{0,y\}}(0) \subset B(0,v_\rho), \mathcal{N}_{\mathbf{X}\cup\{0,y\}}(y) \subset B(y,v_\rho)\right) dy. \quad (4.12)$$

For the first integral, when $y \in \mathfrak{C}_\rho \cap B(0,2v_\rho)^c$, the balls $B(0,v_\rho)$ and $B(y,v_\rho)$ are disjoint. Because $\mathbf{X}$ is a Poisson point process and because $y \notin B(0,2v_\rho)$, the first integrand of (4.12) can be written as the product $\mathbb{P}\left(\mathcal{N}_{\mathbf{X}\cup\{0\}}(0) \subset B(0,v_\rho)\right) \times \mathbb{P}\left(\mathcal{N}_{\mathbf{X}\cup\{y\}}(y) \subset B(y,v_\rho)\right)$. Hence, according to (4.4) and (4.10) we obtain that

$$\mathbb{P}\left(\mathcal{N}_{\mathbf{X}\cup\{0,y\}}(0) \subset B(0,v_\rho), \mathcal{N}_{\mathbf{X}\cup\{0,y\}}(y) \subset B(y,v_\rho)\right) = \mathbb{P}\left(D(\mathcal{C}) < v_\rho\right)^2 \leq c \cdot \rho^{-2}, \quad (4.13)$$

where $y \in \mathbf{R}^d \setminus B(0,2v_\rho)$ and where c is a constant which *does not* depend on y.

For the second integral of (4.12), we apply Lemma 10 to $\chi = \mathbf{X}$. This gives

$$\mathbb{P}\left(\mathcal{N}_{\mathbf{X}\cup\{0,y\}}(0) \subset B(0,v_\rho), \mathcal{N}_{\mathbf{X}\cup\{0,y\}}(y) \subset B(y,v_\rho)\right) \leq \mathbb{P}\left(\#(\mathbf{X} \cap (B(0,v_\rho) \cup B(y,v_\rho)) \geq d+1)\right), \quad (4.14)$$

where $y \in B(0,2v_\rho)$. Since $\#(\mathbf{X} \cap B)$ is Poisson distributed of mean $\lambda_d(B)$ for each Borel subset $B \subset \mathbf{R}^d$, we obtain for ρ large enough that

$$\mathbb{P}\left(\#(\mathbf{X} \cap (B(0,v_\rho) \cup B(y,v_\rho)) \geq d+1)\right) = \sum_{k=d+1}^{\infty} \frac{1}{k!} \left(\lambda_d(B(0,v_\rho) \cup B(y,v_\rho))\right)^k e^{-\lambda_d(B(0,v_\rho)\cup B(y,v_\rho))}$$
$$\leq c \cdot v_\rho^{d(d+1)} = c' \cdot \rho^{-1}$$

according to (4.9) and to the inequalities $e^{-\lambda_d(B(0,v_\rho)\cup B(y,v_\rho))} \leq 1$ and $\lambda_d(B(0,v_\rho) \cup B(y,v_\rho)) \leq 2 \cdot \kappa_d v_\rho^d$, with $y \in B(0,2v_\rho)$. This together with (4.12), (4.13) and (4.14) shows that

$$G_2(\rho) \leq c \cdot \rho^{-1} \lambda_d(\mathfrak{C}_\rho \cap (\mathbf{R}^d \setminus B(0,2v_\rho))) + c \cdot \lambda_d(\mathfrak{C}_\rho \cap B(0,2v_\rho)).$$

Since $\lambda_d(\mathfrak{C}_\rho \cap B(0,2v_\rho)^c) \leq \lambda_d(\mathfrak{C}_\rho) \leq c \cdot \log \rho$ and $\lambda_d(\mathfrak{C}_\rho \cap B(0,2v_\rho)) \leq \lambda_d(B(0,2v_\rho)) = c \cdot \rho^{-1/(d+1)}$, we deduce from the previous inequality that

$$G_2(\rho) \leq c \cdot \log \rho \times \rho^{-1} + c \cdot \rho^{-1/(d+1)} = O\left(\rho^{-1/(d+1)}\right). \quad (4.15)$$

We now derive directly (4.2a) from (4.10), (4.15) and Theorem 1.

Proof of (4.2b). This will be sketched since it is analogous to the proof of (4.2a). First, we investigate the tail of the distribution function of $\lambda_d(\mathcal{F}(\mathcal{C}))$. In [42], Zuyev shows that, conditional on $N_{\mathbf{X}\cup\{0\}} = k$, the volume of $\mathcal{F}(\mathcal{C})$ is Gamma distributed with parameters $(k,1)$, i.e.

$$\mathbb{P}\left(\lambda_d(\mathcal{F}(\mathcal{C})) < v\right) = \sum_{k=d+1}^{\infty} \frac{1}{(k-1)!} \int_0^v x^{k-1} e^{-x} dx \cdot p(k), \quad (4.16)$$

where $p(k) = \mathbb{P}\left(N_{\mathbf{X}\cup\{0\}}(0) = k\right)$. When $k = d+1$, the Taylor expansion $e^{-x} = 1 + O(x)$ shows that the first term of the series in (4.16) equals $\alpha_{d,5} v^{d+1} + O(v^{d+2})$, where

$$\alpha_{d,5} = \frac{p(d+1)}{(d+1)!}. \quad (4.17)$$

If $k \geq d+2$, the term of the series in (4.16) is less than $\frac{1}{d!} \cdot v^{d+2} \cdot p(k)$ thanks to the inequality $e^{-x} \leq 1$. According to (4.16), we get

$$|\mathbb{P}\left(\lambda_d(\mathcal{F}(\mathcal{C})) < v\right) - \alpha_{d,5} \cdot v^{d+1}| = O(v^{d+2}).$$

Hence, for all fixed $t \geq 0$, taking

$$v_\rho = v_\rho(t) = \left(\alpha_{d,5}^{-1}\rho^{-1}\right)^{1/(d+1)} t \tag{4.18}$$

we obtain

$$G_1(\rho) = |\rho\mathbb{P}\left(\lambda_d(\mathcal{F}(\mathcal{C})) < v_\rho\right) - t^{d+1}| = O(\rho^{-1/(d+1)}). \tag{4.19}$$

To get an upper bound for $G_2(\rho)$, we note that for each $\chi \subset \mathbf{R}^d$ locally finite and $x \in \chi$, we have

$$\frac{\kappa_d}{2^d} \cdot \left(D(C_\chi(x))\right)^d \leq \lambda_d\left(\mathcal{F}(C_\chi(x))\right).$$

Applying the previous inequality to $\chi = \mathbf{X} \cup \{0, y\}$ and $x = 0, y$, we deduce from (4.1) that

$$\begin{aligned} G_2(\rho) &= \rho \int_{\mathfrak{C}_\rho} \mathbb{P}\left(\lambda_d(\mathcal{F}(C_{\mathbf{X}\cup\{0,y\}}(0))) < v_\rho, \lambda_d(\mathcal{F}(C_{\mathbf{X}\cup\{0,y\}}(y))) < v_\rho\right) dy \\ &\leq \rho \int_{\mathfrak{C}_\rho} \mathbb{P}\left(D(C_{\mathbf{X}\cup\{0,y\}}(0)) < v'_\rho, D(C_{\mathbf{X}\cup\{0,y\}}(y)) < v'_\rho\right) dy \end{aligned} \tag{4.20}$$

with

$$v'_\rho = v'_\rho(t) = 2\kappa_d^{1/d} \cdot v_\rho^{1/d} = (2^{d(d+1)}\kappa_d^{d+1}\alpha_{d,5}^{-1}\rho^{-1})^{1/d(d+1)} t^{1/d}$$

according to (4.18). Let us notice that there exists a constant c such that, for any $t \geq 0$ and $\rho \geq 0$, we have $v'_\rho(t) = c \cdot v_\rho(t)$, where $v_\rho = v_\rho(t)$ is defined in (4.9). Writing the right-hand side of (4.20) in the same spirit as in (4.11) and proceeding along the same lines as in the proof of (4.2a), we show that $G_2(\rho)$ is of order $\rho^{-1/(d+1)}$. This together with (4.19) shows (4.2b). □

A similar method could be used to re-find the asymptotic behaviour of the minimum of circumradii given in [8].

5 The maximum of inradii of a Gauss-Poisson Voronoi tessellation

As an example of non-Poisson point process, a Gauss-Poisson process is analyzed. Introduced by Newman [24] and investigated by Milne and Westcott, such process has a potential application in statistical mechanics (see [24], p. 350) and could be used as a model for molecular motion (see [21] p. 169). In the sense of [40] p. 161, a stationary planar Gauss-Poisson process $\mathbf{X}$ is a (simple) point process which can be defined as follows. Let $\mathbf{X}_a$ be a Poisson point process of intensity γ_a in $\mathbf{R}^2$. Every point $x_a \in \mathbf{X}_a$ is replaced by a cluster of points $\Xi(x_a) = x_a + \Xi_0(x_a)$, where the sets of points $\Xi_0(x_a)$ are chosen independently and with identical distribution, i.e.

$$\mathbf{X} = \bigcup_{x_a \in \mathbf{X}_a} (x_a + \Xi_0(x_a)). \tag{5.1}$$

For all $x_a \in \mathbf{X}_a$, the cluster $\Xi_0(x_a)$ equals in distribution Ξ_0 which is defined in the following way: Ξ_0 has an isotropic distribution and is composed of zero, one or two points with probability $p_0 \neq 1$, p_1 and $p_2 = 1 - (p_0 + p_1)$.

If Ξ_0 contains only one point then that point is the origin 0. If Ξ_0 is composed of two points then these are separated by a unit distance and have midpoint 0. The intensity of $\mathbf{X}$ is given by $\gamma_{\mathbf{X}} = (p_1 + 2p_2) \cdot \gamma_a$. In this subsection, we investigate the maximum of inradii of a Gauss-Poisson Voronoi tessellation $\mathfrak{m}_{GPVT}$, i.e.

$$r_{\max,GPVT}(\rho) = \max_{x \in \mathbf{X} \cap \mathbf{W}_\rho} r(C_{\mathbf{X}}(x)) \quad \text{where} \quad r(C_{\mathbf{X}}(x)) = \max\{r \geq 0, B(x,r) \subset C_{\mathbf{X}}(x)\}.$$

To apply Theorem 1, we subdivide $\mathbf{W}_\rho$ into $N_\rho = \left\lfloor \left(\frac{\gamma_a(p_1+p_2)\rho}{2\log\rho}\right)^{1/d} \right\rfloor^d$ sub-cubes of equal size. With the same method as for a Poisson-Voronoi tessellation, we can show that there exists an integer $R \geq 1$ and a event A_ρ (in the same spirit as in (3.2)) such that Condition (FRC) holds when the Voronoi tessellation is induced by a Gauss-Poisson process. The asymptotic distribution of $r_{\max}(\rho)$ is given in the following proposition.

Proposition 8. *Let $\mathbf{X}$ be a Gauss-Poisson process of intensity 1, i.e. $(p_1 + 2p_2)\gamma_a = 1$ with $p_0 \neq 1$ and $p_1 \neq 0$. For all $t \in \mathbf{R}$, we have*

$$\left| \mathbb{P}\left(r_{\max,GPVT}(\rho) \leq v_\rho(t)\right) - e^{-e^{-t}} \right| = O\left((\log\rho)^{-1/2}\right),$$

where $v_\rho(t)$ is given in (5.5).

Proof of Proposition 8. We notice that for all $x \in \mathbf{X}$ and $v \geq 0$, the inradius $r(C_{\mathbf{X}}(x))$ is greater than v if and only if $\#\mathbf{X} \cap B(x, 2v) = 1$. Consequently

$$\mathbb{P}\left(r(\mathcal{C}) > v\right) = \mathbb{P}^0(\#\mathbf{X}^0 \cap B(0, 2v) = 1),$$

where $\mathcal{C}$ is the typical cell of the Voronoi tessellation induced by $\mathbf{X}$. In the above equality, $\mathbb{P}^0$ is the Palm measure of $\mathbf{X}$ in the sense of (3.6) of [35] and $\mathbf{X}^0$ is $\mathbb{P}^0$ distributed. The planar Gauss-Poisson process is one of the rare non-Poisson processes for which the right-hand side can be made fully explicit. It is given for each $v \geq 0$ by (see p. 161 in [40]):

$$\mathbb{P}^0(\#\mathbf{X}^0 \cap B(0, 2v) = 1) = \frac{1}{p_1 + 2p_2} e^{-\gamma_a(4p_1\pi v^2 + p_2(8\pi v^2 - a(2v)))} \cdot \begin{cases} p_1 + 2p_2 & 0 \leq 2v < 1 \\ p_1 & 2v \geq 1 \end{cases} \tag{5.2}$$

and

$$a(2v) = 8v^2 \arccos\frac{1}{4v} - \frac{1}{2}\sqrt{16v^2 - 1} \text{ for } 4v \geq 1 \tag{5.3}$$

and equals zero otherwise. The function $a(2v)$ is the area of the intersection of two disks of radius $2v$ and centers separated by unit distance. A Taylor expansion of the right-hand side of (5.2) shows that

$$\mathbb{P}^0(\#\mathbf{X}^0 \cap B(0, 2v) = 1) = e^{-(P(v) + \tilde{R}(v))}, \tag{5.4}$$

where

$$P(v) = 4\gamma_a\pi(p_1 + p_2)v^2 - 4\gamma_a \cdot p_2 \cdot v - \log\left(\frac{p_1}{p_1 + p_2}\right) \text{ and } \tilde{R}(v) = \frac{5\gamma_a \cdot p_2}{48} \cdot \frac{1}{v} + o\left(\frac{1}{v}\right),$$

as v goes to infinity. In the previous line, $\phi(v) = o(\psi(v))$ means that $\phi(v)/\psi(v) \underset{v\to\infty}{\longrightarrow} 0$.

For all $t \in \mathbf{R}$, we define $v_\rho = v_\rho(t)$ so that $P(v_\rho) = \log\rho + t$, i.e.

$$v_\rho = v_\rho(t) = \frac{2\gamma_a \cdot p_2 + \left(4\gamma_a^2 \cdot p_2^2 + 4\gamma_a\pi(p_1 + p_2)\left(\log\left(\frac{p_1}{p_1+2p_2}\right) + \log\rho + t\right)\right)^{1/2}}{4\gamma_a\pi(p_1 + p_2)}. \tag{5.5}$$

Using the fact that $\rho\mathbb{P}^0(\#\mathbf{X}^0 \cap B(0,2v_\rho)=1) = e^{-t-\tilde{R}(v_\rho)}$, we deduce that

$$G_1(\rho) = |\rho\mathbb{P}^0(\#\mathbf{X}^0 \cap B(0,2v)=1) - e^{-t}| \le e^{-t}\tilde{R}(v_\rho) = O\left((\log\rho)^{-1/2}\right). \tag{5.6}$$

To check Condition (LCC), we first provide an integral representation for $G_2(\rho)$ with respect to Palm measures. Then we give some upper bound for the integrand. To do it, we denote by $\mathcal{F}_{lf}$ the space of locally finite subsets of $\mathbf{R}^2$. We deduce from Campbell's theorem (see Theorem 3.3.3. in [35]) that

$$\begin{aligned} G_2(\rho) &= N_\rho \mathbb{E}\left[\sum_{(x,y)\neq\in(\mathbf{X}\cap\mathfrak{C}_\rho)^2} \mathbb{1}_{\#\mathbf{X}\cap B(x,2v_\rho)=1}\mathbb{1}_{\#\mathbf{X}\cap B(y,2v_\rho)=1}\right] \\ &= N_\rho \int_{\mathfrak{C}_\rho}\int_{\mathcal{F}_{lf}} \sum_{y\in\eta\cap\mathfrak{C}_\rho} \mathbb{1}_{\#(\eta+x)\cap B(x,2v_\rho)=1}\mathbb{1}_{\#(\eta+x)\cap B(y,2v_\rho)=1} d\mathbb{P}^0(\eta)dx \\ &= c\cdot\rho \int_{\mathcal{F}_{lf}} \sum_{y\in\eta\cap\mathfrak{C}_\rho} \mathbb{1}_{\#\eta\cap B(0,2v_\rho)=1}\mathbb{1}_{\#\eta\cap B(y,2v_\rho)=1} d\mathbb{P}^0(\eta) \end{aligned}$$

since the integrand of the right-hand side of the second line is translation invariant (in distribution) and $N_\rho\lambda_2(\mathfrak{C}_\rho) = c\cdot\rho$. According to Formula (5.3.2) in [40], we have $\mathbb{P}^0 = \mathbb{P}_\mathbf{X} * \mathbb{c}^0$, where $\mathbb{P}_\mathbf{X}$ is the distribution of $\mathbf{X}$ and $\mathbb{c}^0$ is the Palm measure of the cluster distribution Ξ_0 that is concentrated on the space $\mathcal{F}_{lf,2}$ of subsets of 0, 1 or 2 points in $\mathbf{R}^2$. Hence

$$G_2(\rho) = c\cdot\rho\int_{\mathcal{F}_{lf}}\int_{\mathcal{F}_{lf,2}} \sum_{y\in(\phi\cup\xi)\cap\mathfrak{C}_\rho} \mathbb{1}_{\#(\phi\cup\xi)\cap(B(0,2v_\rho)\cup B(y,2v_\rho))=2}\mathbb{1}_{|y|>2v_\rho} d\mathbb{c}^0(\xi)d\mathbb{P}_\mathbf{X}(\phi).$$

When $|y|>2v_\rho$, we have $y\notin\xi$ for ρ large enough since $\mathbb{c}_0$ a.s. ξ is bounded. Moreover, $\mathbb{P}_\mathbf{X}$ a.s. $\phi\cap\xi\cap(B(0,2v_\rho)\cup B(y,2v_\rho))$ is empty. Consequently, calculating the integral with respect to $\mathbb{c}_0$ and proceeding as previously, we deduce from Campbell's theorem and from the relation $\mathbb{P}^0 = \mathbb{P}_\mathbf{X} * \mathbb{c}^0$ that

$$\begin{aligned} G_2(\rho) &\le c\cdot\rho\int_{\mathcal{F}_{lf}} \sum_{y\in\phi\cap\mathfrak{C}_\rho} \mathbb{1}_{\#\phi\cap(B(0,2v_\rho)\cup B(y,2v_\rho))=1}\mathbb{1}_{|y|>2v_\rho} d\mathbb{P}_\mathbf{X}(\phi) \\ &= c\cdot\rho\int_{\mathfrak{C}_\rho}\int_{\mathcal{F}_{lf}}\int_{\mathcal{F}_{lf,2}} \mathbb{1}_{\#((\xi\cup\phi)+y)\cap(B(0,2v_\rho)\cup B(y,2v_\rho))=1}\mathbb{1}_{|y|>2v_\rho} d\mathbb{c}^0(\xi)d\mathbb{P}_\mathbf{X}(\phi)dy. \end{aligned}$$

Since $\mathbb{P}_\mathbf{X}$ a.s. $\phi\cap\xi\cap(B(0,2v_\rho)\cup B(y,2v_\rho))$ is empty, we deduce after integration over $\mathcal{F}_{lf}\times\mathcal{F}_{lf,2}$ with respect to $\mathbb{P}_\mathbf{X}\otimes\mathbb{c}^0$ that

$$G_2(\rho) \le c\cdot\rho\int_{\mathfrak{C}_\rho} \mathbb{P}\left(\mathbf{X}\cap(B(0,2v_\rho)\cup B(y,2v_\rho))=\varnothing\right)\mathbb{1}_{|y|>2v_\rho}dy. \tag{5.7}$$

We provide below a suitable upper bound for the integrand. Let $|y|>2v_\rho$ be fixed. First, we note that $\mathbf{X}\cap(B(0,2v_\rho)\cup B(y,2v_\rho))=\varnothing$ if and only if $(x+\Xi_0(x))\cap(B(0,2v_\rho)\cup B(y,2v_\rho))=\varnothing$ for all $x\in\mathbf{X}_a$. From Theorem 3.2.4. of [35], Fubini's theorem and the fact that Ξ_0 is symmetric with respect to the origin, we get

$$\begin{aligned} \mathbb{P}\left(\mathbf{X}\cap(B(0,2v_\rho)\cup B(y,2v_\rho))=\varnothing\right) &= e^{-\gamma_a\int_{\mathbf{R}^2}\mathbb{P}((x+\Xi_0(x))\cap(B(0,2v_\rho)\cup B(y,2v_\rho))\neq\varnothing)dx} \\ &= e^{-\gamma_a\mathbb{E}[\lambda_2(\Xi_0+(B(0,2v_\rho)\cup B(y,2v_\rho)))]}. \end{aligned} \tag{5.8}$$

We give below a suitable lower bound for the term appearing in the exponential. Since $|y|>2v_\rho$, we have

$$\mathbb{E}\left[\lambda_2(\Xi_0+(B(0,2v_\rho)\cup B(y,2v_\rho)))|\#\Xi_0=1\right] = \lambda_2\left(B(0,2v_\rho)\cup B(y,2v_\rho)\right) \ge \frac{3}{2}\cdot 4\pi v_\rho^2,$$

$$\mathbb{E}\left[\lambda_2(\Xi_0 + (B(0,2v_\rho) \cup B(y,2v_\rho)))|\#\Xi_0 = 2\right] \geq \mathbb{E}\left[\lambda_2(\Xi_0 + B(0,2v_\rho))\right] \geq 8\pi v_\rho^2 - a(2v_\rho),$$

where $a(\cdot)$ is defined in (5.3). Since Ξ_0 is reduced to 0, 1 or 2 points with probability p_0, p_1 and p_2 respectively, we deduce from (5.8) that

$$\begin{aligned}\mathbb{P}\left(\mathbf{X} \cap (B(0,2v_\rho) \cup B(y,2v_\rho)) = \varnothing\right) &\leq e^{-\gamma_a\left(\frac{3}{2}p_1 \cdot 4\pi v_\rho^2 + p_2(8\pi v_\rho^2 - a(2v_\rho))\right)} \\ &= \frac{p_1 + 2p_2}{p_1}\mathbb{P}^0(\#B(0,2v_\rho) \cap \mathbf{X}^0 = 1) \cdot e^{-2\gamma_a p_1 \pi v_\rho^2}\end{aligned} \tag{5.10}$$

for ρ large enough according to (5.2). Integrating over $\mathfrak{C}_\rho$, we deduce from (5.6), (5.7), (5.10) and from the inequality $\lambda_2(\mathfrak{C}_\rho) \leq c \cdot \log\rho$, that

$$G_2(\rho) \leq c \cdot \log\rho \cdot e^{-2\gamma_a p_1 \pi v_\rho^2} = O\left(\log\rho \cdot \rho^{-\alpha}\right), \tag{5.11}$$

where $\alpha = \frac{p_1}{2(p_1+p_2)}$. Since $p_1 \neq 0$, we have $\alpha > 0$ so that $G_2(\rho)$ converges to 0. Proposition 8 is now a direct consequence of (5.6), (5.11) and Theorem 1.

□

According to Proposition 8 and (5.5), the order of $\mathbb{E}\left[r_{\max,GPVT}(\rho)\right]$ is

$$(4\gamma_a\pi(p_1+p_2))^{-1/2} \cdot (\log\rho)^{1/2} = \left(\frac{p_1 + 2p_2}{4\pi(p_1+p_2)}\right)^{1/2} \cdot (\log\rho)^{1/2}$$

since we have assumed that $(p_1 + 2p_2)\gamma_a = 1$. Let us remark that the larger p_2 is, the larger the order is. This can be explained by the following heuristic fact: the nucleus $x \in \mathbf{X}$ of the Voronoi cell which maximizes the inradius belongs to a cluster of size 1, i.e. $x \in \Xi(x_a)$, where $\#\Xi(x_a) = 1$ for some $x_a \in \mathbf{X}_a$. Hence if p_2 is large, the mean number of clusters of size 1 is small so that the inradii associated with the clusters of size 1 are large.

When $p_1 = 0$, we obtain a degenerate case since $r_{\max,GPVT}(\rho)$ is constant. Indeed, with high probability, the maximum of inradius is made between two points which belong to the same cluster so that $r_{\max,GPVT}(\rho) = \frac{1}{2}$. When $p_0 = p_2 = 0$ and $p_1 = 1$, the random variable $r_{\max,GPVT}(\rho)$ is the maximum of inradii of a Poisson-Voronoi tessellation $r_{\max,PVT}(\rho)$. In that case, the order is

$$v_\rho = v_\rho(t) = (4\pi)^{-1/2} \cdot (\log\rho + t)^{1/2}.$$

The order of $r_{\max,PVT}(\rho)$ has already been investigated in [8]. Nevertheless, Proposition 8 is more precise since it provides a rate of convergence. Actually, this rate could be improved. Indeed, since $p_0 = p_2 = 0$ and $p_1 = 1$ we have $\tilde{R}(v_\rho) = 0$ according to (5.2) and (5.4) and consequently we get $G_1(\rho) = 0$ according to the inequality in (5.6). Moreover, the term α which appears in (5.11) equals 1/2. Hence, according to (5.11), we obtain the more precise result:

$$\mathbb{P}\left(r_{\max,PVT}(\rho) \leq (4\pi)^{-1/2} \cdot (\log\rho + t)^{1/2}\right) = O\left(\log\rho \cdot \rho^{-1/2}\right).$$

Finally, let us mention that a Gauss-Poisson process belongs to the class of the so-called Neyman-Scott processes (see section 5.3 in [40]). We do not investigate general Neyman-Scott processes since we cannot provide a Taylor expansion of $\mathbb{P}\left(r(\mathcal{C}) > r\right)$ in general.

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References

[1] M. Abramowitz and I.A. Stegun. *Handbook of mathematical functions with formulas, graphs, and mathematical tables*, volume 55 of *National Bureau of Standards Applied Mathematics Series*. For sale by the Superintendent of Documents, U.S. Government Printing Office, Washington, D.C., 1964.

[2] R. Arratia, L. Goldstein, and L. Gordon. Poisson approximation and the Chen-Stein method. *Statist. Sci.*, 5(4):403–434, 1990.

[3] F. Avram and D. Bertsimas. On central limit theorem in geometrical probability. *The Annals of Applied Probability*, 3(4):1033–1046, 1993.

[4] F. Baccelli and B. Blaszczyszyn. *Stochastic Geometry and Wireless Networks Volume 2: Applications*. Foundations and Trends in Networking : Vol. 1: Theory, Vol. 2: Applications No 1-2, 2009.

[5] P. Baldi and Y. Rinott. On normal approximations of distributions in terms of dependency graphs. *Ann. Probab.*, 17(4):1646–1650, 1989.

[6] P. Calka. Precise formulae for the distributions of the principal geometric characteristics of the typical cells of a two-dimensional Poisson-Voronoi tessellation and a Poisson line process. *Adv. in Appl. Probab.*, 35(3):551–562, 2003.

[7] P. Calka. Tessellations. In *New perspectives in stochastic geometry*, pages 145–169. Oxford Univ. Press, Oxford, 2010.

[8] P. Calka and N. Chenavier. Extreme values for characteristic radii of a Poisson-Voronoi tessellation. *Available in http://arxiv.org/pdf/1304.0170.pdf*, 2013. To appear in *Extremes*.

[9] D. J. Daley and D. Vere-Jones. *An introduction to the theory of point processes*. Springer Series in Statistics. Springer-Verlag, New York, 1988.

[10] L. de Haan and A. Ferreira. *Extreme value theory. An introduction.* Springer Series in Operations Research and Financial Engineering. Springer, New York, 2006.

[11] L. Heinrich. Normal approximation for some mean-value estimates of absolutely regular tessellations. *Math. Methods Statist.*, 3(1):1–24, 1994.

[12] T. Hsing. On the extreme order statistics for a stationary sequence. *Stochastic Process. Appl.*, 29(1):155–169, 1988.

[13] D. Hug and R. Schneider. Large cells in Poisson-Delaunay tessellations. *Discrete Comput. Geom.*, 31(4):503–514, 2004.

[14] L. Ju, M. Gunzburger, and W. Zhao. Adaptive finite element methods for elliptic PDEs based on conforming centroidal Voronoi-Delaunay triangulations. *SIAM J. Sci. Comput.*, 28(6):2023–2053, 2006.

[15] G. Last, G. Peccati, and M. Schulte. Normal approximation on Poisson spaces: Mehler's formula, second order Poincaré inequalities and stabilization. *Available in http://arxiv.org/pdf/1401.7568v1.pdf*, 2014.

[16] M. R. Leadbetter. On extreme values in stationary sequences. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete*, 28:289–303, 1973/74.

[17] M. R. Leadbetter. Extremes and local dependence in stationary sequences. *Z. Wahrsch. Verw. Gebiete*, 65(2):291–306, 1983.

[18] M. R. Leadbetter, G. Lindgren, and H. Rootzén. *Extremes and related properties of random sequences and processes.* Springer Series in Statistics. Springer-Verlag, New York, 1983.

[19] M. R. Leadbetter and H. Rootzén. On extreme values in stationary random fields. In *Stochastic processes and related topics*, Trends Math., pages 275–285. Birkhäuser Boston, Boston, MA, 1998.

[20] R. E. Miles. A synopsis of "Poisson flats in Euclidean spaces". *Izv. Akad. Nauk Armjan. SSR Ser. Mat.*, 5(3):263–285, 1970.

[21] R.K. Milne and M. Westcott. Further results for Gauss-Poisson Process. *Adv. in Appl. Probab.*, 4(1):151–176, 1972.

[22] J. Møller. Random tessellations in $\mathbf{R}^d$. *Adv. in Appl. Probab.*, 21(1):37–73, 1989.

[23] J. Møller. *Lectures on random Voronoi tessellations*, volume 87 of *Lecture Notes in Statistics.* Springer-Verlag, New York, 1994.

[24] D. S. Newman. A new family of point processes which are characterized by their second moment properties. *J. Appl. Probab.*, 7:338–358, 1970.

[25] A. Okabe, B. Boots, K. Sugihara, and S. N. Chiu. *Spatial tessellations: concepts and applications of Voronoi diagrams.* Wiley Series in Probability and Statistics. John Wiley & Sons Ltd., Chichester, second edition, 2000.

[26] M. Penrose. *Random geometric graphs*, volume 5 of *Oxford Studies in Probability.* Oxford University Press, Oxford, 2003.

[27] M. D. Penrose and J. E. Yukich. Normal approximation in geometric probability. In *Stein's method and applications*, volume 5 of *Lect. Notes Ser. Inst. Math. Sci. Natl. Univ. Singap.*, pages 37–58. Singapore Univ. Press, Singapore, 2005.

[28] M. B. Petrovskaya and A. M. Leontovich. The central limit theorem for a sequence of random variables with a slowly growing number of dependences. *Teor. Veroyatnost. i Primenen.*, 27(4):757–766, 1982.

[29] A. Poupon. Voronoi and Voronoi-related tessellations in studies of protein structure and interaction. *Current Opinion in Structural Biology*, 14(2):233–241, 2004.

[30] M. Ramella, W. Boschin, D. Fadda, and M. Nonino. Finding galaxy clusters using Voronoi tessellations. *Astronomy and Astrophysics*, 368:776–786, 2001.

[31] P. N. Rathie. On the volume distribution of the typical Poisson-Delaunay cell. *J. Appl. Probab.*, 29(3):740–744, 1992.

[32] S. I. Resnick. *Extreme values, regular variation, and point processes*, volume 4 of *Applied Probability. A Series of the Applied Probability Trust.* Springer-Verlag, New York, 1987.

[33] W. L. Roque. Introduction to Voronoi Diagrams with Applications to Robotics and Landscape Ecology. *Proceedings of the II Escuela de Matematica Aplicada*, 01:1–27, 1997.

[34] W. Schaap. *DTFE: the Delaunay Tessellation Field Estimator.* PhD Thesis, 2007.

[35] R. Schneider and W. Weil. *Stochastic and integral geometry.* Probability and its Applications (New York). Springer-Verlag, Berlin, 2008.

[36] M. Schulte and C. Thäle. The scaling limit of Poisson-driven order statistics with applications in geometric probability. *Stochastic Process. Appl.*, 122(12):4096–4120, 2012.

[37] M. Shamos and D. Hoey. Closest–point problems. *16th Annual Symposium on Foundations of Computer Science*, pages 151–162, 1975.

[38] R. L. Smith. Extreme value theory for dependent sequences via the Stein-Chen method of Poisson approximation. *Stochastic Process. Appl.*, 30(2):317–327, 1988.

[39] M. Spanel, P. Krsek, M. Svub, V. Stancl, and O. Siler. Delaunay-Based Vector Segmentation of Volumetric Medical Images. *Computer Analysis of Images and Patterns*, 4673:261–269, 2007.

[40] D. Stoyan, W.S. Kendall, and J. Mecke. *Stochastic Geometry and its applications.* Wiley, 2008.

[41] H. Zessin. Point processes in general position. *J. Contemp. Math. Anal., Armen. Acad. Sci.*, 43(1):59–65, 2008.

[42] S. A. Zuyev. Estimates for distributions of the Voronoi polygon's geometric characteristics. *Random Structures Algorithms*, 3(2):149–162, 1992.